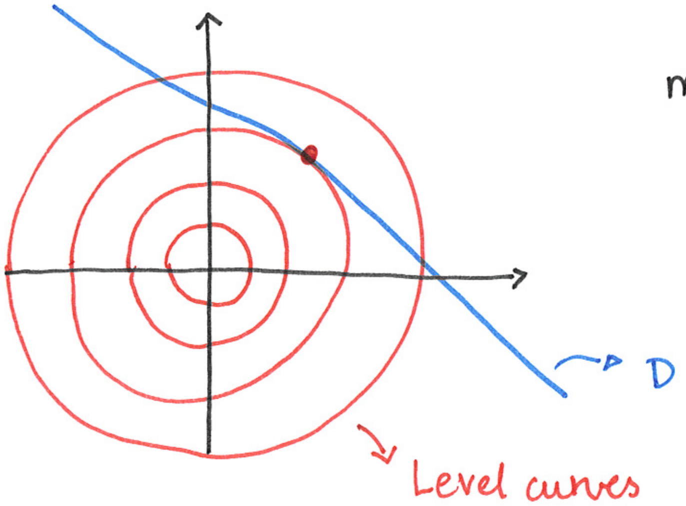
Lagrange problems ctd.

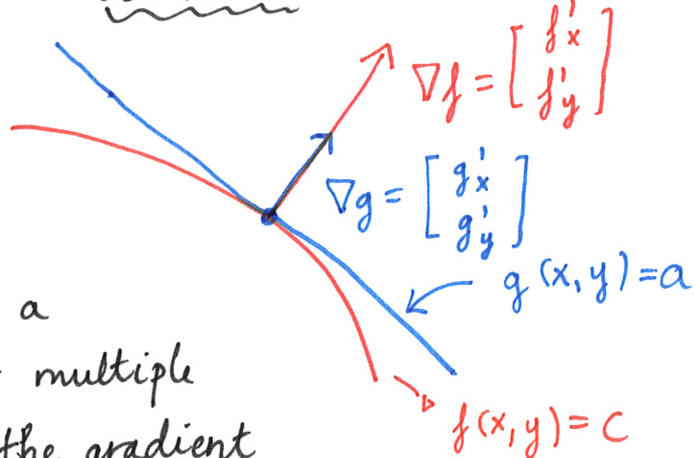
EBA 1180
Lect. 4 6/22
S26



$$\max/\min \quad \overbrace{x^2+y^2}^{f(x,y)}$$

$$\text{when} \quad \underbrace{\overbrace{3x+4y}^{g(x,y)} = 12}_{D}$$

Zoom in:



NOTE: $\nabla f = \lambda \nabla g$ ← Gradient of f is a scalar multiple of the gradient of g

$$\begin{bmatrix} f'_x \\ f'_y \end{bmatrix} = \lambda \begin{bmatrix} g'_x \\ g'_y \end{bmatrix}$$

$$\begin{cases} f'_x = \lambda g'_x \\ f'_y = \lambda g'_y \end{cases} \Rightarrow \begin{cases} \mathcal{L}'_x = f'_x - \lambda g'_x = 0 \\ \mathcal{L}'_y = f'_y - \lambda g'_y = 0 \end{cases}$$

Theorem: If (x^*, y^*) is a max/min. in a Lagrange problem:

$$\boxed{\max/\min \ f(x,y) \quad \text{with} \quad g(x,y) = a}$$

Then, either:

i) There is a λ s.t. $(x^*, y^*; \lambda)$ satisfy the Lagrange constraints FOC + C:

FOC: $\begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases}$ and C: $g(x,y) = a$

OR

ii) The constraint is degenerate at (x^*, y^*), i.e.:

$$g'_x = 0 \quad \text{and} \quad g'_y = 0 \quad \text{and} \quad g(x,y) = a$$

Ex: In general, an extreme point (max/min) with a degenerate constraint is a point where D doesn't have a unique tangent:

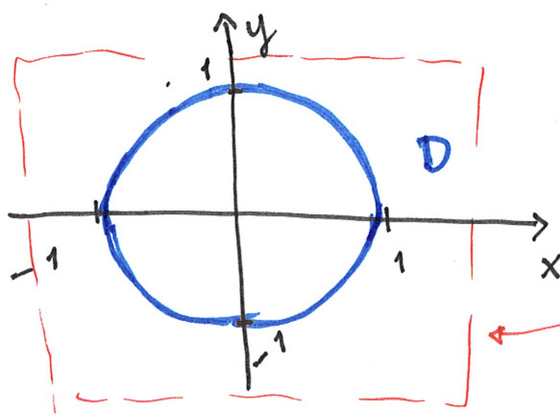
Ex ctd. from last time: $g(x,y) = 3x + 4y$

Degenerate constraint? $g'_x = 3 \neq 0$

$g'_y = 4 \neq 0$

so case ii) of Theorem is not possible.

Ex: max/min $f(x,y) = xy$ when $\underbrace{x^2 + y^2 = 1}_{D}$ Circle, center $(0,0)$, $r = \sqrt{1} = 1$



D is compact: • Closed (=)

• Bounded

f is continuous $\Rightarrow$ EVT: f has a max and min over D.

Type ii) candidates: Degenerate constraint?

$$\begin{cases} g'_x = 0 \\ g'_y = 0 \end{cases} \iff \begin{cases} 2x = 0 \iff x = 0 \\ 2y = 0 \iff y = 0 \end{cases}$$

But then: $x^2 + y^2 = 0^2 + 0^2 = 0 \neq 1$, so the constraint doesn't hold. Hence, there are no admissible points with degenerate constraint $\Rightarrow$ No type ii) candidates.

NB: Holds in general for circles.

Type i) candidates:

Lagrangian: $\mathcal{L}(x, y; \lambda) = xy - \lambda(x^2 + y^2 - 1)$

$$\boxed{\begin{array}{ll} \text{FOC:} & \mathcal{L}'_x = y - \lambda \cdot 2x = 0 \\ & \mathcal{L}'_y = x - \lambda \cdot 2y = 0 \\ \text{C:} & x^2 + y^2 = 1 \end{array}}$$

3 eqns. with 3 unknowns: x, y, λ

$$y = 2\lambda x$$

$$x - 2\lambda \cdot 2\lambda x = 0$$

$$x(1 - 4\lambda^2) = 0$$

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$x = 0$:

$y = 2\lambda x = 2\lambda \cdot 0 = 0$

C: $x^2 + y^2 = 0^2 + 0^2 = 0 \neq 1$

$\Rightarrow$ The constraint doesn't hold

$1 - 4\lambda^2 = 0$:

$\lambda^2 = \frac{1}{4}$

$\lambda = \frac{1}{2}$:

$\lambda = -\frac{1}{2}$:

$\Rightarrow$ Not a candidate point.

$\underset{\sim\sim\sim}{\lambda = \frac{1}{2}}$:

$y = 2 \cdot \frac{1}{2} x = x$

$\underset{\sim}{C}$: $x^2 + y^2 = 1$

$x^2 + x^2 = 1$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}} = y$

Candidate points:

$\underbrace{\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)}, \underbrace{\left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)}$

$f(x,y) = xy$

$= \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}} = \underline{\frac{1}{2}}$ HERE:

$f = \underline{\frac{1}{2}}$

$\boxed{\text{MAX POINTS}}$

Know f has max. over D from EVT.

$\underset{\sim\sim\sim}{\lambda = -\frac{1}{2}}$:

$y = 2\left(-\frac{1}{2}\right)x = -x$

$\underset{\sim}{C}$: $x^2 + y^2 = 1$

$x^2 + (-x)^2 = 1$

$x^2 + x^2 = 1$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

$y = -x$

Candidate points:

$\left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right), \underbrace{\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)}$

$f = \sqrt{\frac{1}{2}}\left(-\sqrt{\frac{1}{2}}\right)$

$= -\frac{1}{2}$

$f = -\frac{1}{2}$

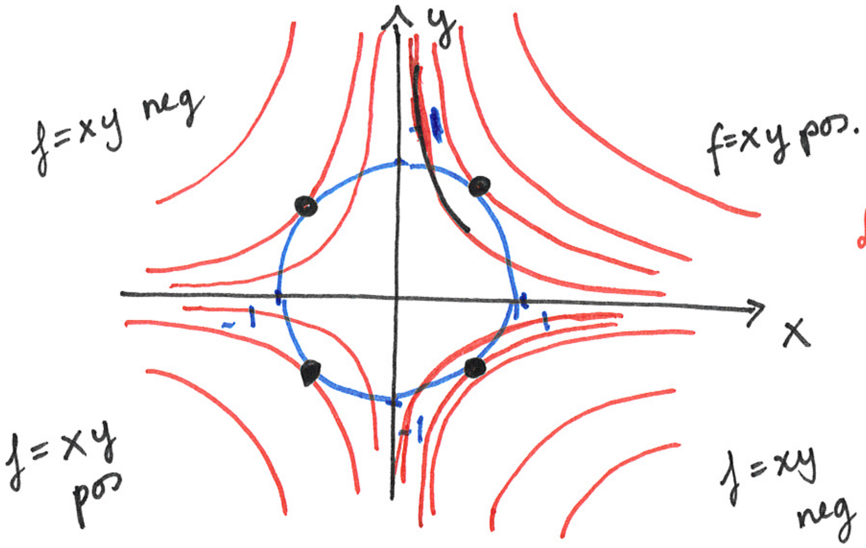
$\boxed{\text{MIN POINTS}}$

Know f has min. over D from EVT.

Makes sense? $f(x,y) = xy$

Level curves: $xy = c, \; c \neq 0$

$y = \frac{c}{x} \rightarrow$ Hyperbolas



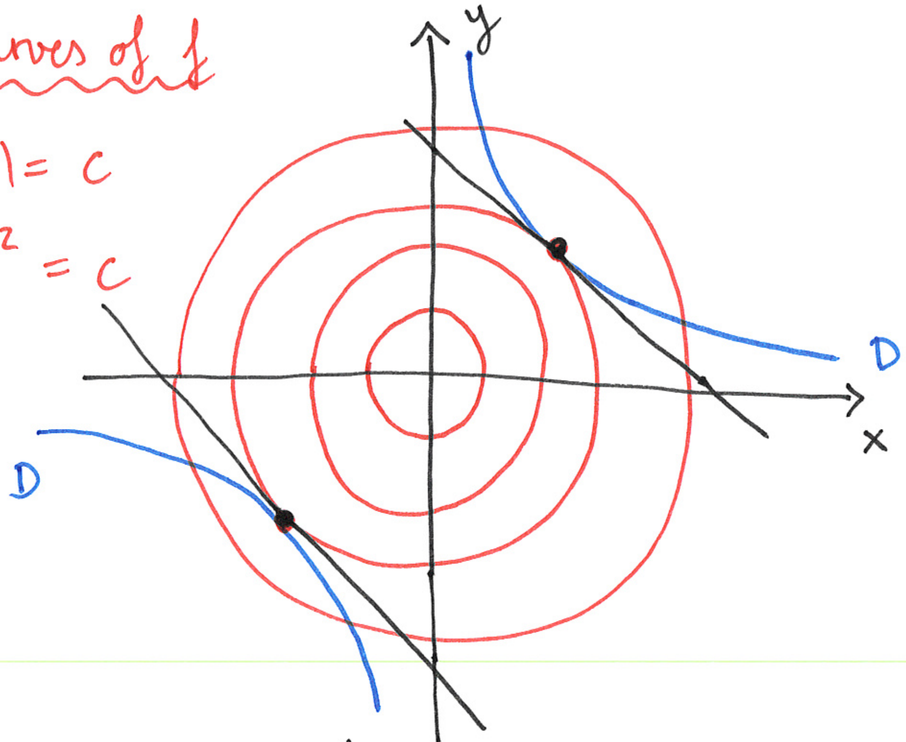
Level curves of $f = xy$

Ex: max/min $f(x,y) = x^2 + y^2$ when $\underbrace{xy = 1}$

Level curves of f

$f(x,y) = c$

$x^2 + y^2 = c$



D: $\frac{1}{x} = y$,

x can't be 0 since then $xy = 0$

EVT? Closed? ✓

Bounded? No! Can't box in constraint curve D.

f continuous? ✓

$\Rightarrow$ Can't use EVT.

Type i) candidates: $\mathcal{L}(x,y;\lambda) = x^2 + y^2 - \lambda(xy - 1)$

FOC:

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 2x - \lambda y = 0 \\ \mathcal{L}'_y &= 2y - \lambda x = 0 \\ xy &= 1 \end{aligned}}$$

C:

3 equations with 3 unknowns: x, y, λ