Lagrange problems

Optimization problem (max/min) with <u>equality constraints</u>.

$$\boxed{\max/\min \; f(x,y) \quad \text{when} \quad \underbrace{g(x,y)}_{\text{function}} = \underset{\text{constant}}{a}} \qquad (\star)$$

Ex:

$$\min \; f(x,y) = x^2 + y^2 \quad \text{when} \quad \underset{\text{equality constraint}}{3x + 4y = 12}$$

A Lagrange problem.

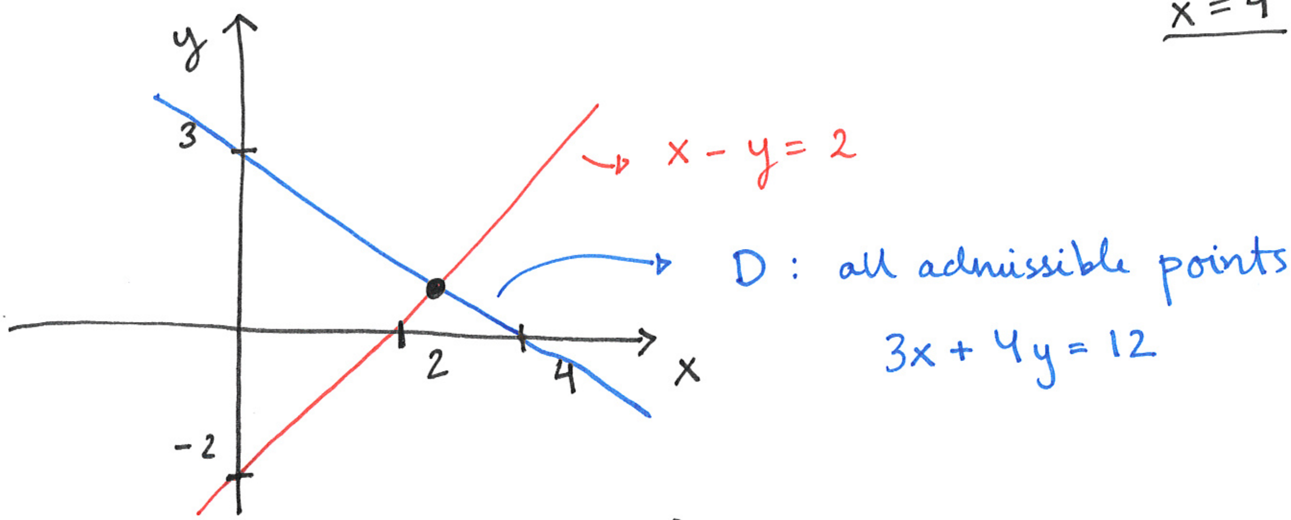
Draw: $4y = 12 - 3x$

$y = 3 - \frac{3}{4}x$

$x = 0$: $y = 3 - 0 = 3$

$y = 0$: $3x = 12$

$x = 4$



Ex: $\min \; f(x,y) = x^2 + y^2$ when $3x + 4y = 12$
and $\boxed{x - y = 2}$ } A point: Intersection of two str. lines

$\Downarrow$

$y = x - 2$

Recap: General method

1) Find candidate points: i) Interior stationary pts: NONE

ii) Other interior critical points: NONE (No interior, only boundary)

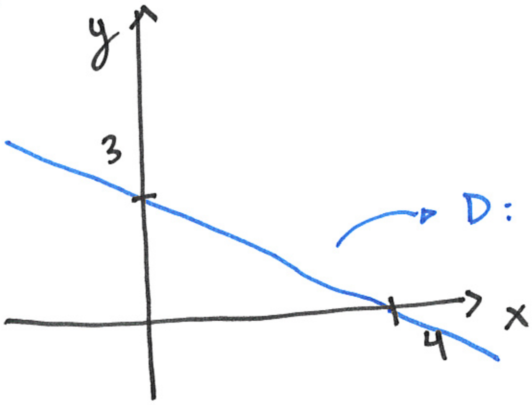
iii) Boundary points: All admissible points.

2) Determine whether any of these are max / min: Always true for Lagrange problems: $=$ const.

Extreme value theorem? If D is compact (closed and bounded) and if f is continuous, then f has a max and min over D.

Not necessarily true for Lagrange problems

Ex ctd.: $\min f(x,y) = x^2 + y^2$
when $3x + 4y = 12$



D: $3x + 4y = 12$

D is closed (=), but not bounded

$\Downarrow$

D is not compact $\Rightarrow$ EVT can't be used.

Method of Lagrange multipliers

"Lambda"

$$\underbrace{\mathcal{L}(x,y;\lambda)}_{\text{Lagrangian (Lagrange function)}} = f(x,y) - \lambda\,(\underbrace{g(x,y) - a}_{})$$

Lagrange multiplier: Variable

NOTE: $=0$ if the constraint in (*) holds.

$$= x^2 + y^2 - \lambda(3x + 4y - 12)$$

Example

Candidates for max/min: The stationary pts. of $\mathcal{L}$:

$$\text{FOC: } \begin{cases} \mathcal{L}'_x = f'_x - \lambda g'_x = 0 = 2x - 3\lambda \\ \mathcal{L}'_y = f'_y - \lambda g'_y = 0 = 2y - 4\lambda \end{cases}$$

$$\text{C: } \{ \mathcal{L}'_\lambda = -(g(x,y) - a) = 0 = -(3x + 4y - 12)$$

$$g(x,y) - a = 0 \qquad \Updownarrow$$

The constraint $\rightarrow \boxed{g(x,y) = a} \longleftrightarrow \boxed{3x + 4y = 12}$

Lagrange conditions: FOC + C

Ex: FOC:

$$\begin{aligned} \mathcal{L}'_x &= 2x - 3\lambda = 0 \quad (1) \\ \mathcal{L}'_y &= 2y - 4\lambda = 0 \quad (2) \end{aligned}$$

C:

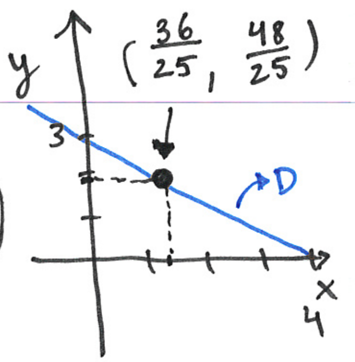
$$3x + 4y = 12 \quad (3)$$

⇝ System of 3 eqns. with 3 unknowns: x, y, λ

$$\begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -4 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

To solve: Gaussian elimination (or isolating variables + substituting)

DIY ⇝ $\lambda = \frac{24}{25}$, $x = \frac{36}{25}$, $y = \frac{48}{25}$



Only one candidate: $\left(\underbrace{\frac{36}{25}}_{x}, \underbrace{\frac{48}{25}}_{y} ; \underbrace{\frac{24}{25}}_{\lambda} \right)$

START: 11.05

Alternative method : Substitution

$$\boxed{\min \quad f(x,y) = x^2 + y^2 \quad \text{when} \quad 3x + 4y = 12}$$

$$y = 3 - \frac{3}{4}x \qquad (\blacktriangle)$$

$$x^2 + y^2 = x^2 + \left(3 - \frac{3}{4}x\right)^2$$

$$= x^2 + 9 - \frac{9}{2}x + \frac{9}{16}x^2$$

$$= \frac{25}{16}x^2 - \frac{9}{2}x + 9 =: g(x)$$

Alt:

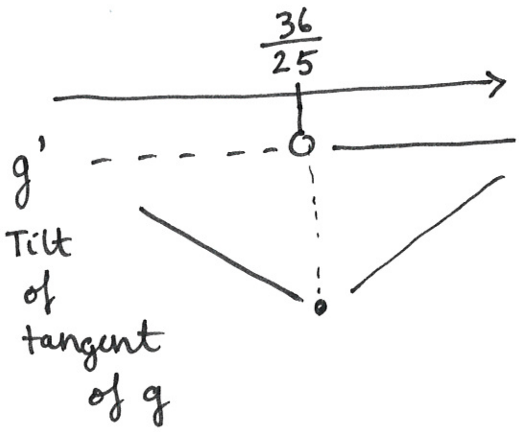
$$\boxed{\min \quad g(x) = \frac{25}{16}x^2 - \frac{9}{2}x + 9}$$

An unconst. one-variable optimization problem

$$g'(x) = \frac{25}{8}x - \frac{9}{2} = 0 \quad \Big| \cdot 8$$

$$25x - 36 = 0$$

$$x = \frac{36}{25}$$



From $(\blacktriangle)$:

$$y = 3 - \frac{3}{4} \cdot \frac{36}{25} = \ldots = \frac{48}{25}$$

Hence, $x = \frac{36}{25}$ is a minimum for g.

Intuition: Lagrange multiplier method

Ex: max/min $f(x,y) = x^2 + y^2$ when $\underbrace{3x + 4y}_{g(x,y)} = 12$

Level curves of f: $f(x,y) = c$

$$x^2 + y^2 = c$$

- $c > 0$: Circle, center $(0,0)$, $r = \sqrt{c}$
- $c = 0$: A point $(0,0)$
- $c < 0$: No level curve

$c=1$: $x^2+y^2=1$, $r=\sqrt{1}=1$

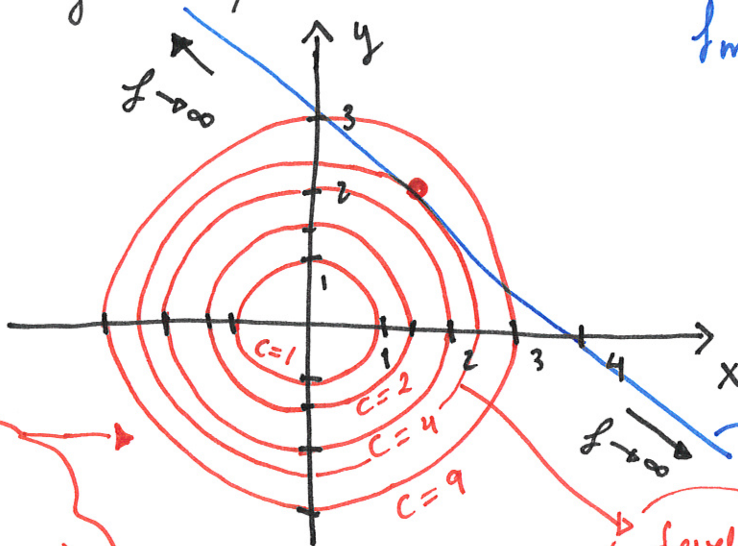
$c=2$: $x^2+y^2=2$, $r=\sqrt{2}$

$c=4$: $x^2+y^2=4$, $r=\sqrt{4}=2$

$c=9$: $x^2+y^2=9$, $r=\sqrt{9}=3$

$f_{min} = \left(\frac{36}{25}\right)^2 + \left(\frac{48}{25}\right)^2$

$= 5{,}76$



Level curves of $f(x,y)$

D: $\underbrace{3x + 4y}_{g(x,y)} = 12$

Level curve of f at level 5,76

Zoom in:

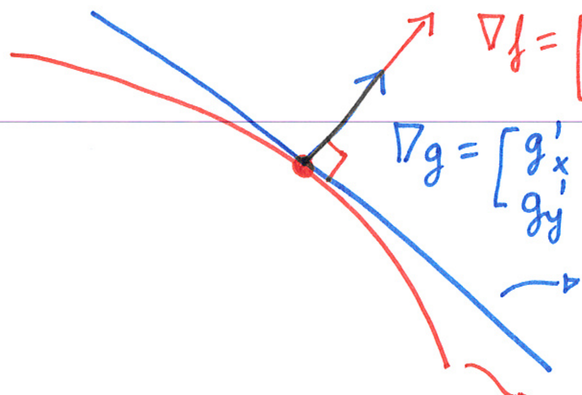


Fig. 1

$\nabla f = \begin{bmatrix} f'_x \\ f'_y \end{bmatrix}$

$\nabla g = \begin{bmatrix} g'_x \\ g'_y \end{bmatrix}$

$g(x,y) = a$;

$f(x,y) = c$

NOTE: D is also a level curve for g at level a

Candidates for max/min:

Points where the two curves meet at a tangent: Slopes of tangents of level curves should be equal:

$$-\frac{f'_x}{f'_y} = -\frac{g'_x}{g'_y}$$

See note on tangents of level curves: Implicit diff.

Ex. ctd:

$$-\frac{2x}{2y} = -\frac{3}{4}$$

$\vdots$ (solve)

$$y = \frac{4}{3}x$$

Constraint: $3x + 4y = 3x + 4 \cdot \frac{4}{3}x = 12$

$\vdots$ (solve)

$$\underline{\underline{x = \frac{36}{25}}}$$

NOTE: $\nabla f = \lambda \nabla g$ $\leadsto$ From Fig. 1: Gradient of f is a scalar multiple of the gradient of g