Constrained optimization

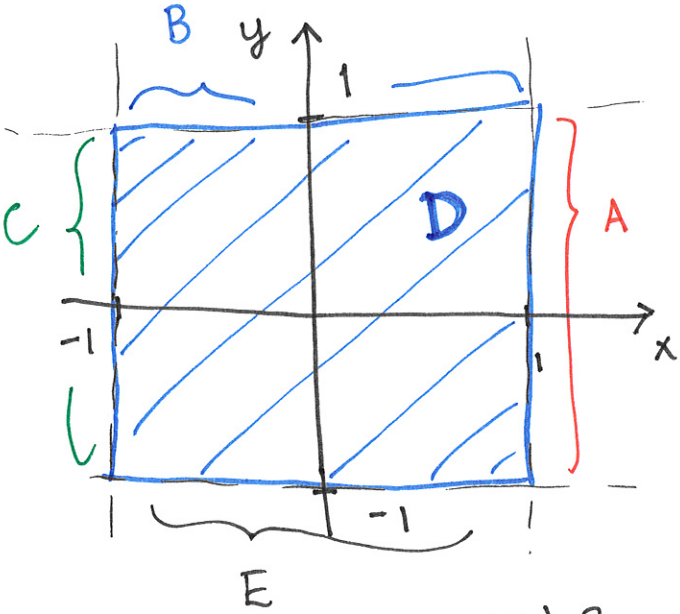
Ctd. from last time:

EBA 1180
Sect. 20/44
S26

Ex: max/min $f(x,y) = x^2 + y^2$ when

$\underbrace{-1 \leq x, y \leq 1}$

D = all (x,y) such that constraints hold



Candidate points

i) Interior Stationary pts: $(0,0)$ → $f(0,0) = 0^2 + 0^2 = 0$

ii) Interior critical pts: None

iii) Boundary: A: $(1,1)$, $(1,-1)$, $\underbrace{(1,0)}$

$f(1,1) = 1^2 + 1^2 = 2 = f(1,-1)$

$f(1,0) = 1^2 + 0^2 = 1$

B: $y=1, -1 \leq x \leq 1$: $f(x,1) = x^2 + 1$

C:

E: Repeat! (or symmetry) $(-1,1)$, $(-1,-1)$, $\underbrace{(-1,0), (0,-1), (0,1)}$

$f(-1,1) = f(-1,-1) = 2$

$f(-1,0) = f(0,-1) = f(0,1) = 1$

Extreme value thm.

- f continuous? Yes.
- D compact: → Closed? Yes.
  → Bounded? Yes.

EVT holds ⇒ $f(x,y)$ has a max. and min. over D.

Conclusion

There is a max. and a min. from EVT. The largest function value among the candidates is:

$$\boxed{f_{max} = 2}$$

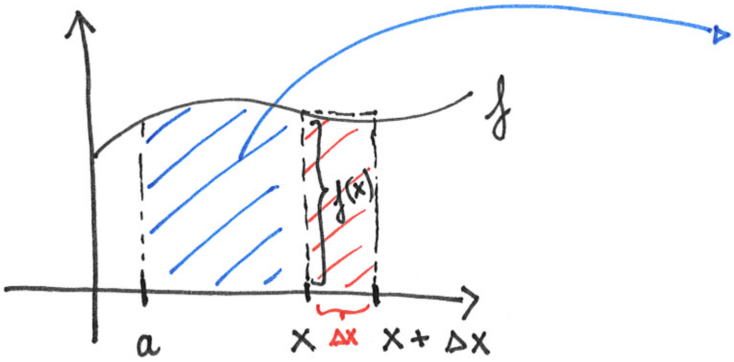
at the max. points: $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$

The smallest value among the candidates is:

$$\boxed{f_{min} = 0}$$

at the min. point $(0, 0)$.

Repetition interlude: Fundamental thm. of calculus



A(x): Area under graph of f from a to x.

$$A(x) = \int_a^x f(t)\,dt$$

From figure: Red area

$$A(x+\Delta x) - A(x) \approx f(x)\,\Delta x$$

$$A'(x) \approx \frac{A(x+\Delta x) - A(x)}{\Delta x} \approx f(x)$$

Def. of derivative

Can be made precise (no $\approx$). So:

$$A'(x) = \boxed{\left(\int_a^x f(t)\,dt \right)' = f(x)}$$

Def. of A(x)

Examples: Optimization

Ex: $f(x,y) = x^2y^3 + y^2 - 2y$, $D_f = \mathbb{R}^2$ } Unconstrained optimization

$\boxed{\text{No boundary}}$

Q: What's the plan to max/min $f(x,y)$?

$\boxed{\text{START: 11.00}}$

Candidate points

i) Stationary points:

$$f'_x = 2xy^3 = 0 \Rightarrow x = 0 \text{ or } y = 0$$

$$f'_y = 3x^2y^2 + 2y - 2 = 0$$

Can both x and y be 0?

$3 \cdot 0^2 \cdot 0^2 + 2 \cdot 0 - 2 = -2 \neq 0$

2 cases:

$x = 0$	$y = 0$
$3 \cdot 0^2 \cdot y^2 + 2y - 2 = 0$	$3x^2 \cdot 0^2 + 2 \cdot 0 - 2 = 0$
$2y = 2$	$-2 = 0$
$y = 1$	Not true $\Rightarrow (x, 0)$ not possible.

Hence, only one stationary point $(x^*, y^*) = (0, 1)$.

$$f(0,1) = 0^2 \cdot 1^3 + 1^2 - 2 \cdot 1 = -1$$

ii) other critical points: Partial derivatives are defined everywhere $\Rightarrow$ No other critical points.

iii) Boundary: No boundary.

$\Rightarrow$ The stationary point is the only candidate point.

Classification (local):

$$H(f)(x,y) = \begin{bmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{bmatrix}$$

f''_{xx} f''_{xy} f''_{yx} f''_{yy}

Insert $(0,1)$:

$$H(f)(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det H(f)(0,1) = 2\cdot 2 - 0\cdot 0 = 4 > 0,$$

$$\operatorname{tr} H(f)(0,1) = 2+2 = 4 > 0$$

$\Downarrow$ 2nd derivative test

$(0,1)$ is a local minimum: $f(0,1) = -1$

Recall: $f(x,y) = x^2y^3 + y^2 - 2y$

Conclusion (problem specific):

If $x=1$, then:

$$f(1,y) = 1^2y^3 + y^2 - 2y = \underbrace{y^3 + y^2 - 2y}$$

Can be made arbitrarily large: The y^3+y^2 will overpower $-2y$ for large y's. Try: $y = 100, 1000, 10\,000 \Rightarrow$

$\boxed{\text{No global max.}}$

Can also be made arbitrarily small: The y^3 will overpower y^2-2y for very negative y's. Try: $y=-100, -1000, -10000$

really negative

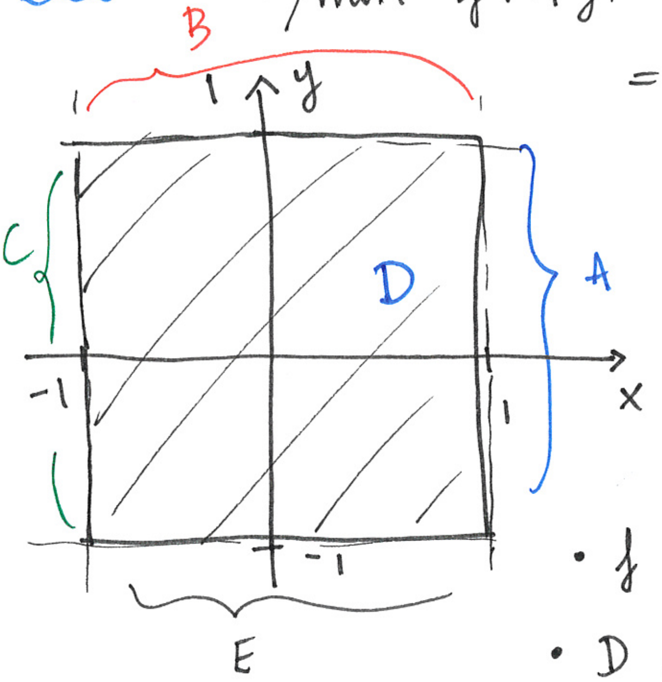
$\Rightarrow$ $\boxed{\text{No global min.}}$

Example of a function with one stationary point, but no global min.

local min.

Ex: max/min $f(x,y) = xy(x^2 - y^2)$

$= x^3y - xy^3$, when



$$\underbrace{-1 \leq x, y \leq 1}_{D}$$

Extreme value theorem?

- f continuous? Yes.
- D compact: → closed? Yes.

  → Bounded? Yes

$\Downarrow$ EVT holds!

f has a max. and a min. over D.

Inside D

Candidate points

i) Interior stationary points: $f'_x = 3x^2y - y^3 = 0$

$f'_y = x^3 - 3xy^2 = 0$

(1) $y(3x^2 - y^2) = 0$

(2) $x(x^2 - 3y^2) = 0$

From (1):

$y=0$ or $3x^2 - y^2 = 0 \rightarrow \boxed{3x^2 = y^2}$

(2): $x(x^2 - 3\cdot 0^2) = 0$

$x^3 = 0$

$\underline{x = 0}$

(2): $x(x^2 - 3\cdot 3x^2) = 0$

$x(x^2 - 9x^2) = 0$

$-8x^3 = 0$

$\underline{x = 0}$

$y^2 = 3x^2 = 3\cdot 0^2 = 0 \Rightarrow$

$\underline{y = 0}$

Only one interior stationary pt. $(x,y) = (0,0)$:

$$\underline{f(0,0) = 0}$$

ii) Other interior critical pts: f'_x and f'_y are def. everywhere $\Rightarrow$ No such pts.

iii) Boundary of D: A, B, C, E $\rightsquigarrow$ DIY on exercise sheet !