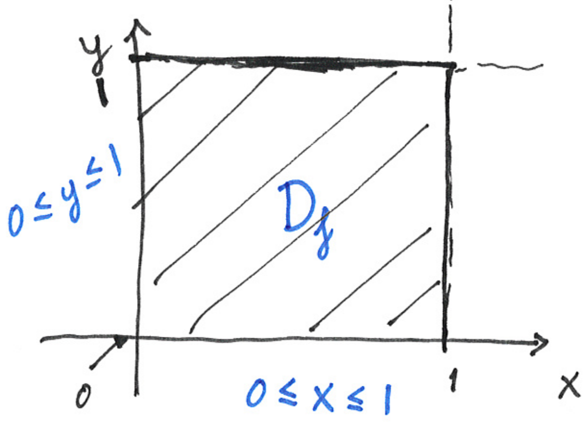
Constrained optimization and the extreme value theorem

EBA 1180
Sect. 4.3
S. 26

Warm-up: $f(x,y) = x + y, \quad \underbrace{0 \le x, y \le 1}_{D_f}$

i.e. $0 \le x \le 1$ and $0 \le y \le 1$



Q: What are the (global) max./min. points?

Max: $(1,1) \Rightarrow f(1,1) = 1+1 = \underline{2}$ (make x and y as big as possible to make sum big)

Min: $(0,0) \Rightarrow f(0,0) = 0+0 = \underline{0}$ (make x and y as small as possible to make sum small)

NB: f has both (global) min. and max.

Q: What if $D_f: \ 0 < x, y < 1$? No max/min.

Q: What if min/max $f(x,y) = x+y$ over all of $\mathbb{R}^2$? No max/min.

Assume: $f(x,y)$ is a continuous function on a set D in $\mathbb{R}^2$.

"EVT"

EXTREME VALUE THEOREM: If f is a continuous function on a compact set D in $\mathbb{R}^2$, then f has a maximum and a minimum on D.

Compact sets

Def (compact set): A subset D of $\mathbb{R}^2$ is compact if it is closed and bounded.

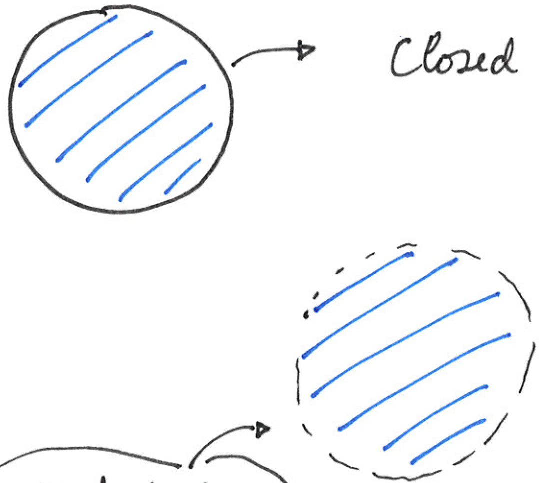
Def (closed set): A subset D of $\mathbb{R}^2$ is closed if all boundary points of D are included in D.

NOTE: $=$, $\leq$, $\geq$; closed

$<$, $>$; not closed

Ex:

Solid line means pts. are included



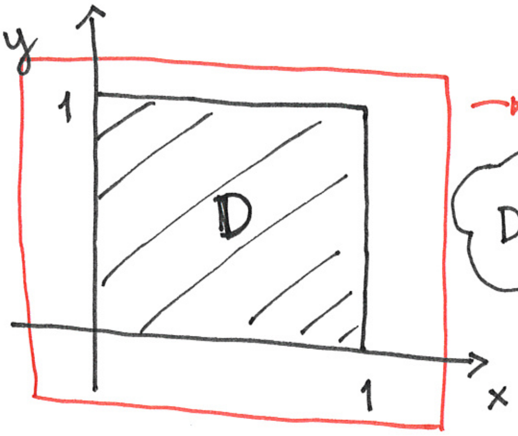
Closed (boundary included)

Not closed (boundary not included)

Dashed line means pts. are not included

Def (bounded set): A subset D of $\mathbb{R}^2$ is bounded if there exists a rectangle in $\mathbb{R}^2$ (with finite side lengths) that includes all of D.

Ex:



→ Rectangle that includes all of D

$D: 0 \le x, y \le 1$

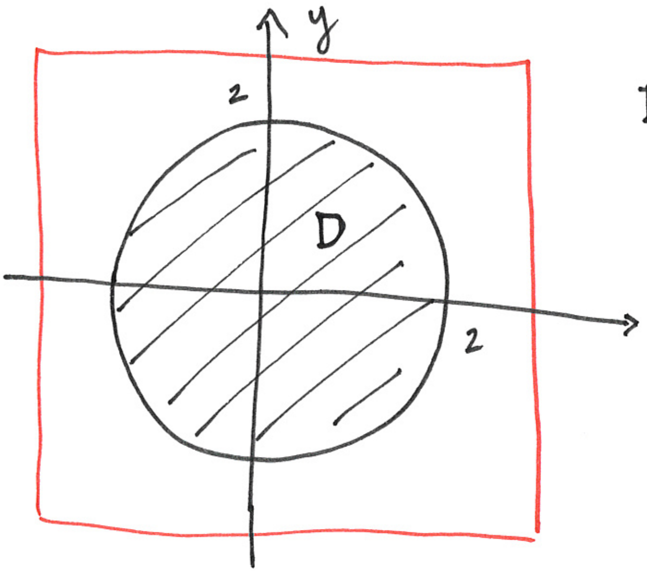
D is:

- bounded? ✓
- closed? ✓

⇙

Compact? ✓

Ex:



$D: \quad x^2 + y^2 \le 4$

(filled circle)

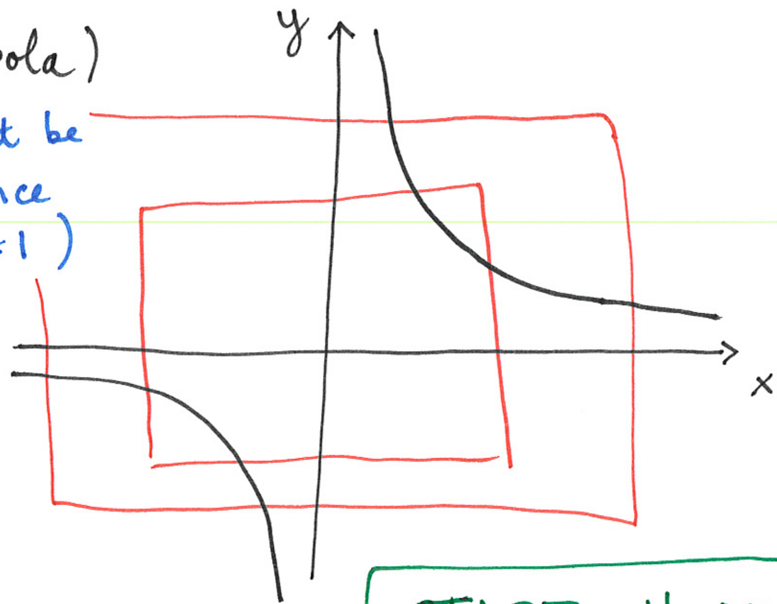
Q:
- bounded? ✓
- closed? ✓

⇙

Compact? ✓

Ex: $D: \; xy = 1$ (hyperbola)

$y = \frac{1}{x}$ (x can't be 0 since $xy=1$)



Q:
- bounded? NO!
- closed? ✓

Compact? NO!

START: 11.04

$$\max_D f(x,y) = x^2 + y^2 = \underbrace{x^2 + \frac{1}{x^2}}$$

→ ∞ as $x \to \infty$ ⇒ No max.

Constrained optimization

max/min $f(x,y) = x^2 + y^2$ when $0 \le x, y \le 1$

objective function

constraints

$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x, y \le 1\}$, subset of $\mathbb{R}^2$

Set of admissible points

Unconstrained optimization

max/min $f(x,y) = x^2 + y^2$

METHOD

Unconstrained

Candidate pts:

i) Interior Stationary pts: $f'_x = 0$, $f'_y = 0$

ii) Other critical pts: f'_x OR f'_y are not def.

iii) Boundary pts. of D_f.

LOCAL CLASSIFICATION:

Second derivative test

(local max, local min, saddle pt)

Must check: Are any of these global max/min.

Problem specific

Constrained

Candidate pts:

i) Interior stationary pts: $f'_x = 0$, $f'_y = 0$

ii) Other critical pts: f'_x OR f'_y not defined

iii) Boundary pts. of D → ∂D; the boundary of D

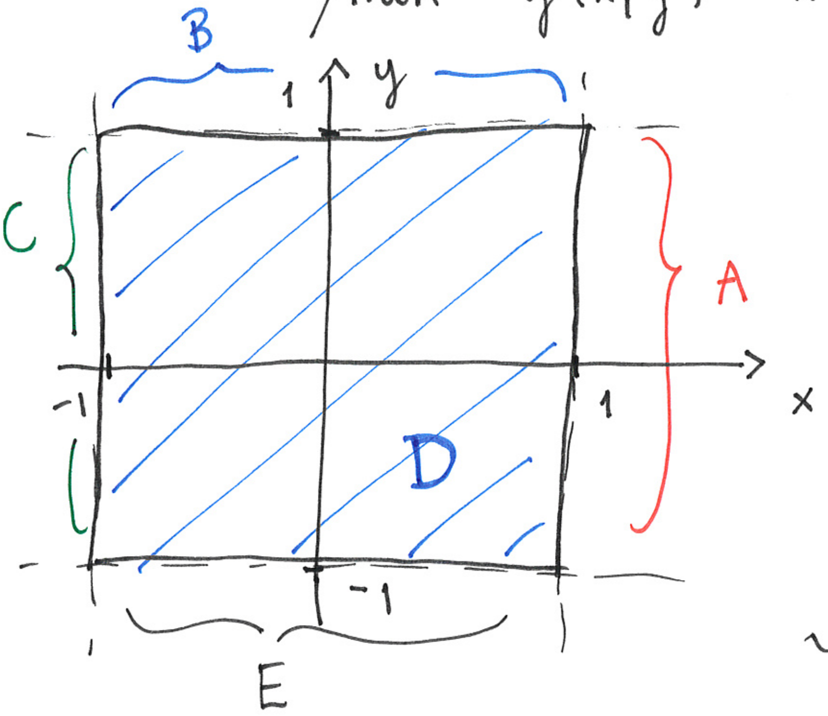
$D = \{(x,y) :$ all constraints are satisfied$\}$

(LOCAL CLASSIFICATION of interior pts.)

EVT: If D is compact & f cont., there is a global max and min.

Can find max/min by comparing function values in candidate pts.

Ex: max/min $f(x,y) = x^2 + y^2$ when



$-1 \le x, y \le 1$

$D = \{(x,y) \in \mathbb{R}^2 : -1 \le x, y \le 1\}$

$\boxed{\text{CONSTRAINED OPTIMIZATION}}$

Candidate pts:

i) Interior stationary points: $f'_x = 0$, $f'_y = 0$

$$\left.\begin{array}{lll} f'_x = 2x = 0 & \Rightarrow & x = 0 \\ f'_y = 2y = 0 & \Rightarrow & y = 0 \end{array}\right\} (0,0); \text{ an interior point of } D$$

Candidate: $(x,y) = (0,0) \Rightarrow f(0,0) = 0^2 + 0^2 = 0$

ii) Other interior critical pts: No such pts.

iii) Boundary pts. of D: ∂D = four sides of square

A: $x = 1$, $-1 \le y \le 1$

B: $y = 1$, $-1 \le x \le 1$

C: $x = -1$, $-1 \le y \le 1$

E: $y = -1$, $-1 \le x \le 1$

1) A : $f(1, y) = 1^2 + y^2 = \underbrace{1 + y^2}$, $-1 \leq y \leq 1$

x always 1 on A

Func. in one variable : 1 variable optimization: Diff., set equal zero & compare with end pts

Max : $f(1,1) = f(1,-1) = 1^2 + 1^2$
$= 1 + 1^2$
$= 2$

Min : $f(1,0) = 1 + 0^2 = 1$

Repeat for B, C and E.