# Tangents of level curves ctd.



**Ex:** $f(x,y) = x^2 - 2x + y^2 + 4y$

**Last time:** Found level curves: All $(x,y)$ s.t.

$$f(x,y) = c$$

**3 cases:**

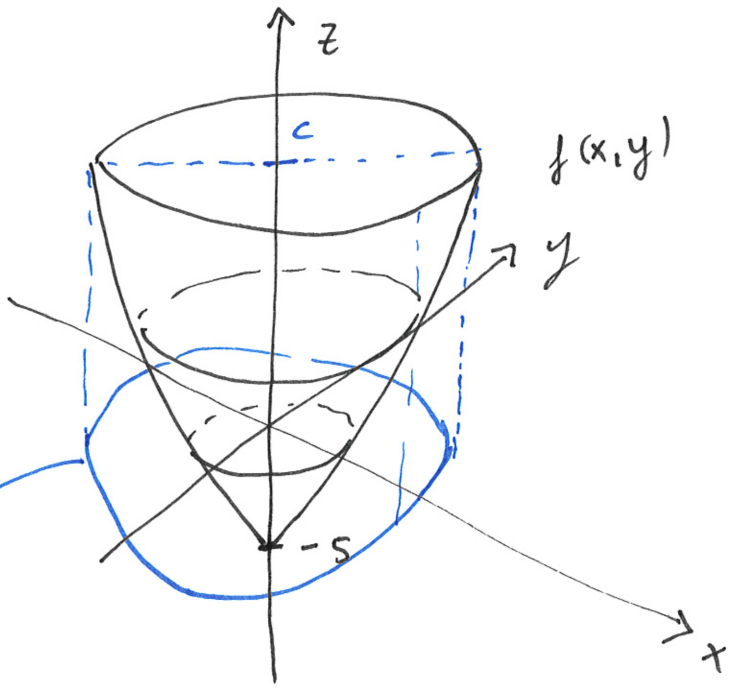
**$c > -5$:** Circle, center $(1,-2)$, $r = \sqrt{c+5}$

**$c = -5$:** $(1,-2)$

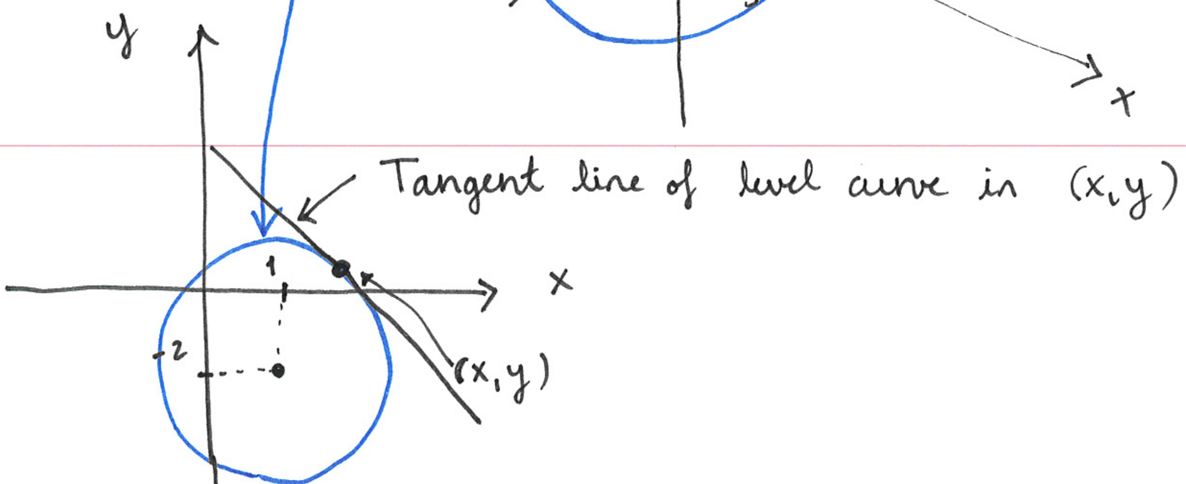
**$c < -5$:** No level curve

How to find tangent line of a level curve in a point $(x,y)$?

**Where are we?**




Level curve:



Ex. ctd.: Tangent line of level curve of $f$ in $(x,y) = (-2, 2)$?

$$f(x,y) = x^2 - 2x + y^2 + 4y$$

1) Find level of level curve corresponding to the point:

$$z = f(-2,2) = (-2)^2 - 2\cdot(-2) + 2^2 + 4\cdot 2$$

$$= 4 + 4 + 4 + 8 = 20$$

$$\Rightarrow \quad c = 20$$

Level curve at level 20 corresp. to $(x,y) = (-2,2)$.

2) Point-slope formula: $y - 2 = k\,(x - (-2))$

$$y = k\,(x+2) + 2 \qquad (\sim)$$

What is the slope $k$?

3) Implicit differentiation: Think $y = y(x)$. At level curve:

$$f(x,y) = c$$

$$(x^2 - 2x + y^2 + 4y)'_x = (20)'_x$$

$$2x - 2 + \underbrace{2y(x)\,y'(x)}_{\text{chain rule}} + 4y'(x) = 0$$

$$2x - 2 + 2y\,y' + 4y' = 0$$

Solve for $y'$:

$$y' = -\frac{2x-2}{2y+4} \boxed{= -\frac{f'_x}{f'_y}}$$

OBS!

Why? $y'$ = the slope, since $y$ is lin. function of $x$:

$$y = \boxed{k}x + b$$

$$y' = \boxed{k}$$

4) Insert $(x,y) = (-2,2)$:

$$y'\Big|_{(-2,2)} = -\frac{2\cdot(-2)-2}{2\cdot 2+4} = \frac{3}{4}$$

the slope of the tangent line in $(-2,2)$

Hence, from $(\sim)$:

$$y = \frac{3}{4}(x+2)+2$$

$$= \frac{3}{4}x + \frac{3}{2} + 2 = \frac{3}{4}x + \frac{7}{2}$$

The tangent line of the level curve of $f(x,y)$ in $(-2,2)$

RESULT: If $f(x,y) = c$, then

$$f'_x + f'_y\, y' = 0 \rightarrow$$ Implicit differentiation + chain rule (as above)

Hence, $\underbrace{y'}_{\text{slope}} = -\dfrac{f'_x}{f'_y}$

# Interpretation of partial derivatives

What does $f'_x(a,b)$ and $f'_y(a,b)$ mean?

**Ex:** $f(x,y) = x^3 - 3xy + y^3$

$\Rightarrow f'_x(x,y) = 3x^2 - 3y \quad , \quad f'_y = -3x + 3y^2$

Let $(x,y) = (2,1)$. Then:

$$f(2,1) = 2^3 - 3 \cdot 2 \cdot 1 + 1^3 = \underline{3}$$

$$f'_x(2,1) = 3 \cdot 2^2 - 3 \cdot 1 = \underline{9}$$

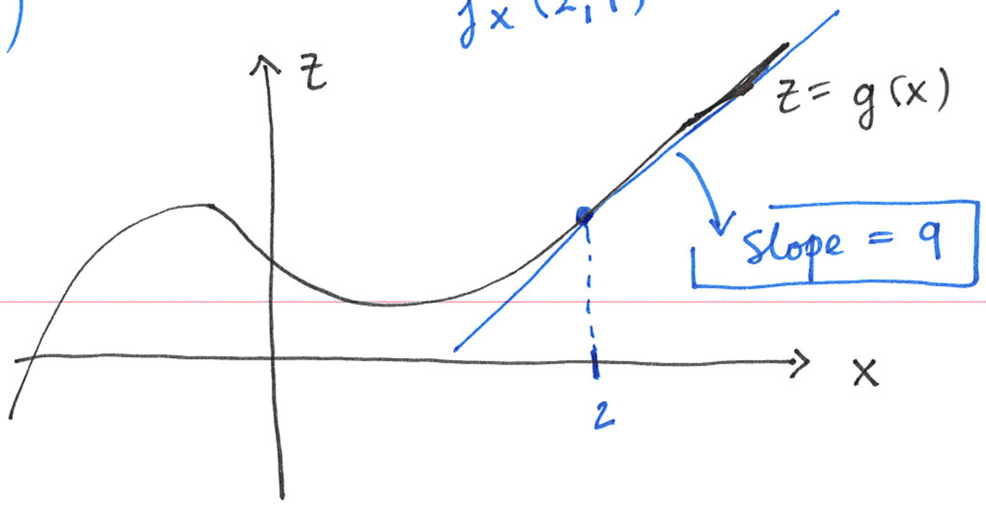
SAME

$$f'_y(2,1) = -3 \cdot 2 + 3 \cdot 1^2 = \underline{-3}$$

In the x-direction, $y = 1$:

$$f(x,1) = x^3 - 3 \cdot x \cdot 1 + 1^3 = x^3 - 3x + 1 \overset{\text{Def. as}}{=:} g(x)$$

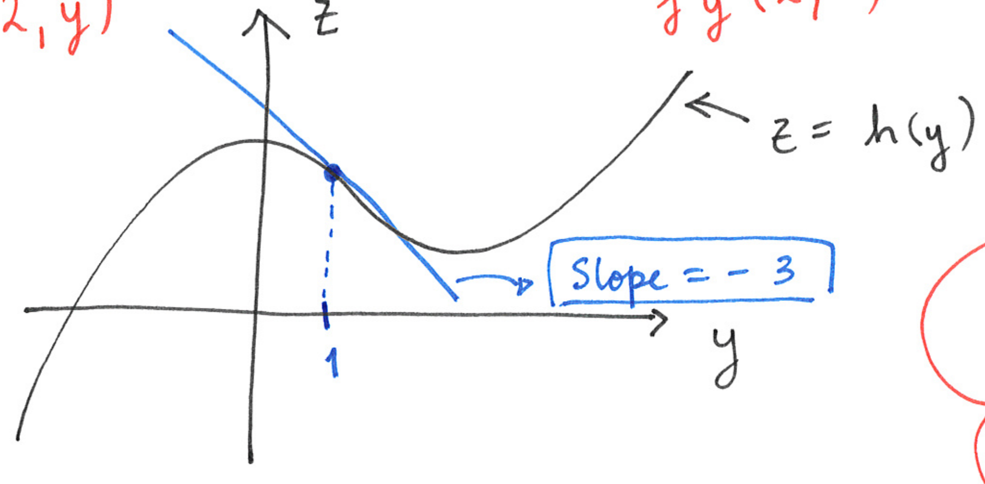
$$\underbrace{g'(x)}_{f'_x(x,1)} = 3x^2 - 3 \quad , \quad \underbrace{g'(2)}_{f'_x(2,1)} = 3 \cdot 2^2 - 3 = \underline{9}$$



In the y-direction, $x = 2$:

$$f(2,y) = 2^3 - 3\cdot 2\cdot y + y^3 = 8 - 6y + y^3 =: h(y)$$

$$\Rightarrow \underbrace{h'(y)}_{= f'_y(2,y)} = -6 + 3y^2, \text{ so } \underbrace{h'(1)}_{= f'_y(2,1)} = -6 + 3\cdot 1^2 = \underline{-3}$$




START
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Linear approximation to $f(x,y)$ at $(x_0,y_0)$:

Tangent plane of $f$ at $(x_0,y_0)$

$$L(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)(x - x_0) + f'_y(x_0,y_0)(y - y_0)$$

Recall: Plane: $c + \underbrace{a}_{f'_x(x_0,y_0)}x + \overbrace{b}^{f'_y(x_0,y_0)}y$

# The gradient

**Def (gradient):** The gradient of $f(x,y)$ is

$$\underbrace{\nabla f}_{\text{"The gradient of } f\text{"}} = \underbrace{\begin{bmatrix} f'_x \\ f'_y \end{bmatrix}}_{\text{a vector}}$$

**Ex:** $f(x,y) = x^2 - 2x + y^2 + 4y$

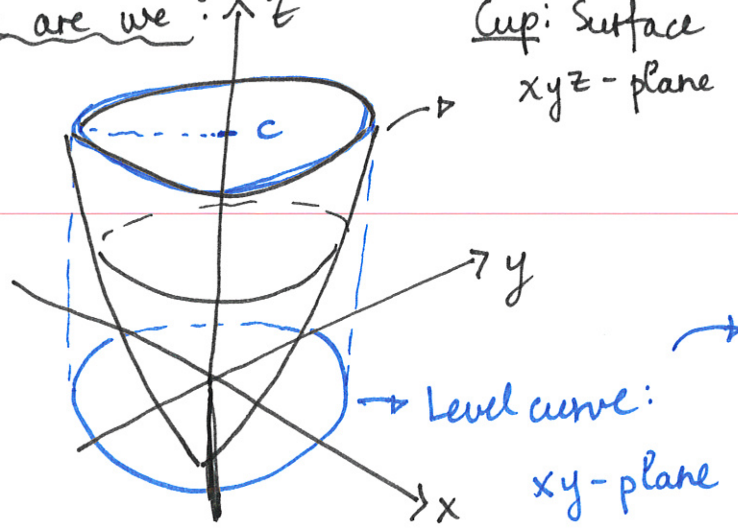
$$\nabla f = \begin{bmatrix} 2x - 2 \\ 2y + 4 \end{bmatrix}$$

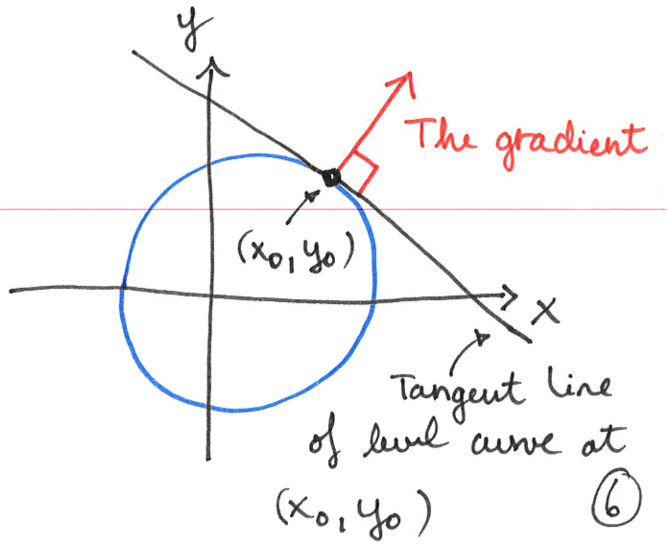
The gradient of $f$ in $(x,y) = (-2,2)$:

$$\nabla f(-2,2) = \begin{bmatrix} 2\cdot(-2) - 2 \\ 2\cdot 2 + 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$$

**RESULT:** The gradient at a point is a normal vector to the tangent line of the level curve at that point.

Where are we?



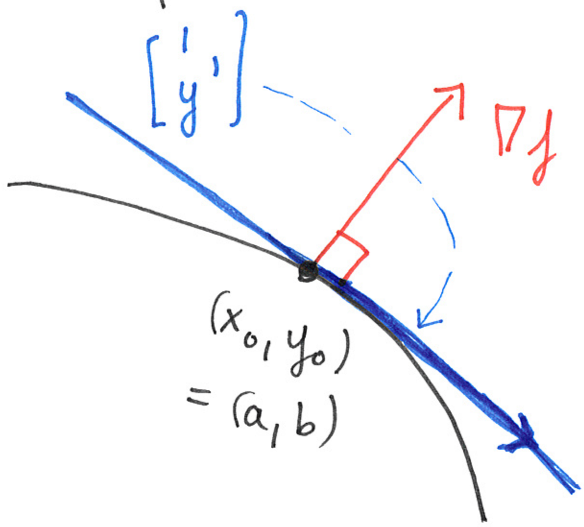
Why does the result hold? $\quad (f(x,y))_x = (C)_x$

Implicit differentiation: $\quad f'_x + f'_y\, y' = 0$

$\boxed{\text{wrt. } x : \; y = y(x)}$

Write as inner product:

$$\underbrace{\begin{bmatrix} 1 \\ y' \end{bmatrix}}_{\substack{\text{Vector in direction} \\ \text{of tangent of} \\ \text{level curve}}} \cdot \underbrace{\begin{bmatrix} f'_x \\ f'_y \end{bmatrix}}_{\nabla f} = 0$$




$f(x,y) = c$

So: $\nabla f(a,b)$ is a vector which has a $90°$ angle on the tangent of the level curve at the point $(a,b)$.

Ex: $\quad \nabla f(-2,2) = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$ $\quad$ earlier today

Tangent line of level curve of $f$ at $(-2,2)$:

$$y = \frac{3}{4}x + \frac{7}{2}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{3}{4}x + \frac{7}{2} \end{bmatrix} = \begin{bmatrix} x \\ \frac{3}{4}x \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{7}{2} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}_{\substack{\text{Vector in the direction} \\ \text{of the tangent line of level curve}}} x + \begin{bmatrix} 0 \\ \frac{7}{2} \end{bmatrix}$$

of $f$ at $(-2,2)$.

$$\nabla f(-2,2) \cdot \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = -6 + \cancel{8}^{\,2} \cdot \frac{3}{\cancel{4}}$$

$$= -6 + 6 = \underline{0}$$

<u>So:</u> $\nabla f(-2,2) \perp \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}$, i.e., the tangent line of the level curve at $(-2,2)$.

## Directional derivative

**Def (Directional derivative):** Let $f(x,y)$ be a function, $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ a 2-vector. Then,

$$\underbrace{f'_{\vec{a}}}_{\text{"The directional derivative of } f \text{ wrt. } \vec{a}\text{"}} := \vec{a} \cdot \nabla f$$

"The directional derivative of $f$ wrt. $\vec{a}$"

dot product: a number

**Ex:** $f(x,y)$ as before, $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$f'_{\vec{a}} = \vec{a} \cdot \nabla f = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2x-2 \\ 2y+4 \end{bmatrix}$$

$$= 2(2x-2) + 1(2y+4)$$

$$= \ldots = 4x + 2y \rightarrow \text{a number!}$$

Can insert a point: Say $(1,1)$: $f'_{\vec{a}}(1,1) = 4 \cdot 1 + 2 \cdot 1 = \underline{\underline{6}}$

THURSDAY: Course paper 2023: TRY yourself