Graphs and level curves

EBA 1180
Lect. 14
S 26

Def (level curve): All (x,y) s.t.
$$f(x,y) = c \quad , \text{ for a constant } c.$$

Ex: $f(x,y) = x^2 + y^2$

Level curves:

$c=2$: $f(x,y) = 2$
$x^2 + y^2 = 2 \rightarrow$ Circle, center $(0,0)$, $r = \sqrt{2}$

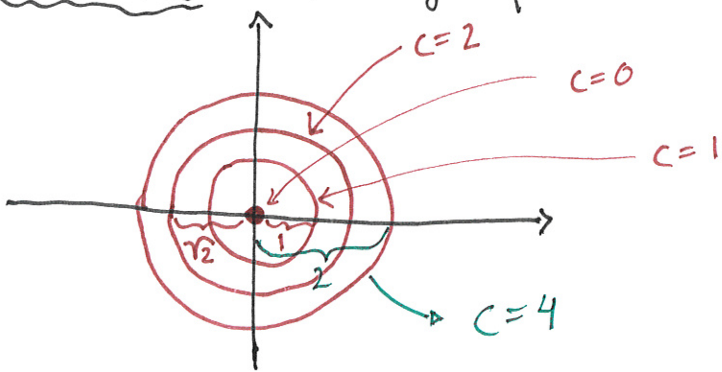
$c=1$: $f(x,y) = 1$
$x^2 + y^2 = 1 \rightarrow$ Circle, center $(0,0)$, $r = \sqrt{1} = 1$

$c=0$: $f(x,y) = 0$
$x^2 + y^2 = 0 \Leftrightarrow x = y = 0$; level "curve" is just a point

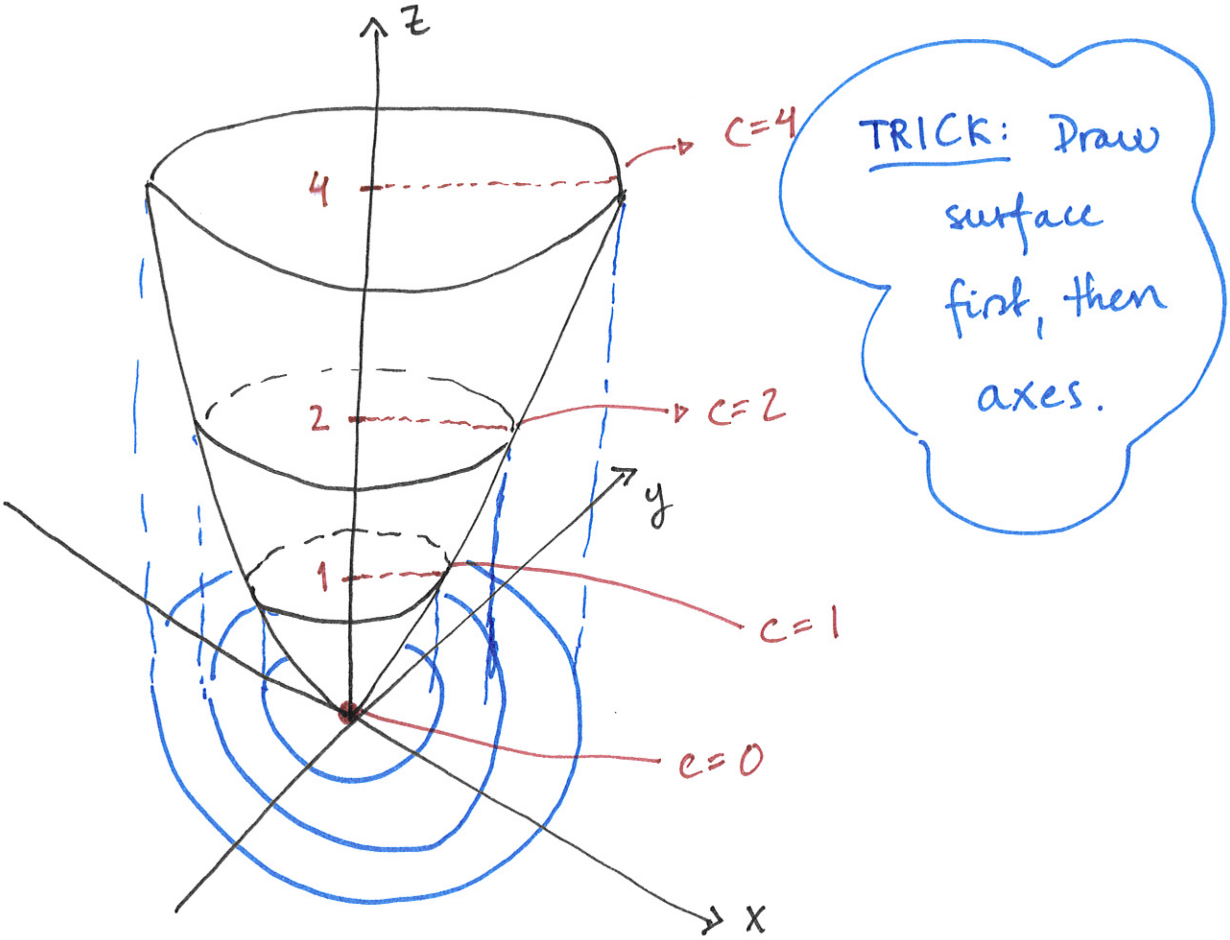
$c=4$: $f(x,y) = 4$
$x^2 + y^2 = 4 \rightarrow$ Circle, center $(0,0)$, $r = \sqrt{4} = 2$

$c=-1$:
$f(x,y) = -1$
$x^2 + y^2 = -1 \rightarrow$ No such points

Illustration of level curves: In xy-plane



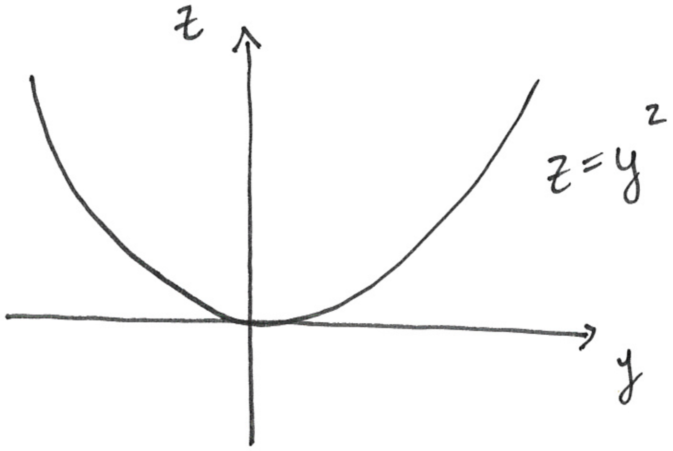
Use level curves to draw graph of f:



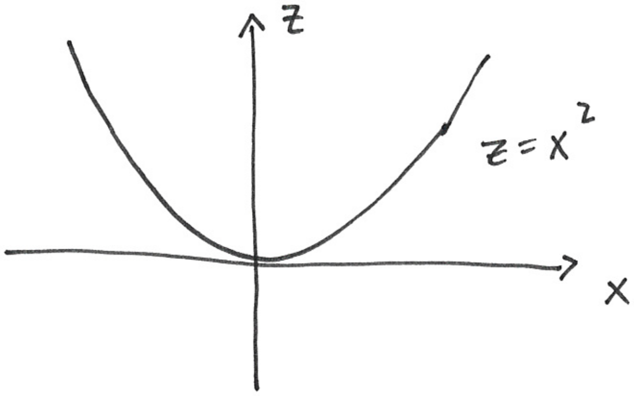
Q: What does $z = f(x,y)$ look like if $x=0$?

Then; $f(0,y)$

Cut $x=0$: $z = f(0,y) = 0^2 + y^2 = y^2$



Cut $y=0$: $\quad z = f(x,0) = x^2 + 0^2 = x^2$



Linear functions

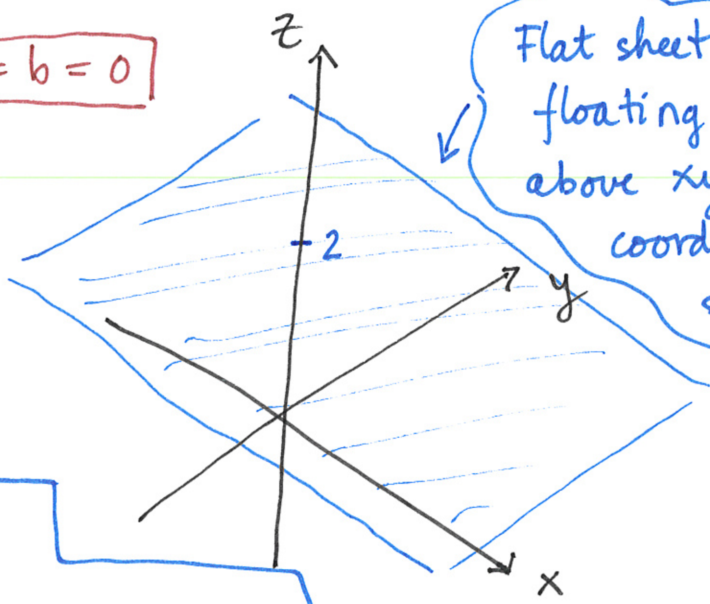
Def (Linear function): A function in two variables is linear if it can be written:

$$f(x,y) = ax + by + c$$

FACT: The graph of f is a plane $\Leftrightarrow$ f is linear.

Ex: $f(x,y) = 2$ $\quad \boxed{a=b=0}$

$z = f(x,y) = 2$



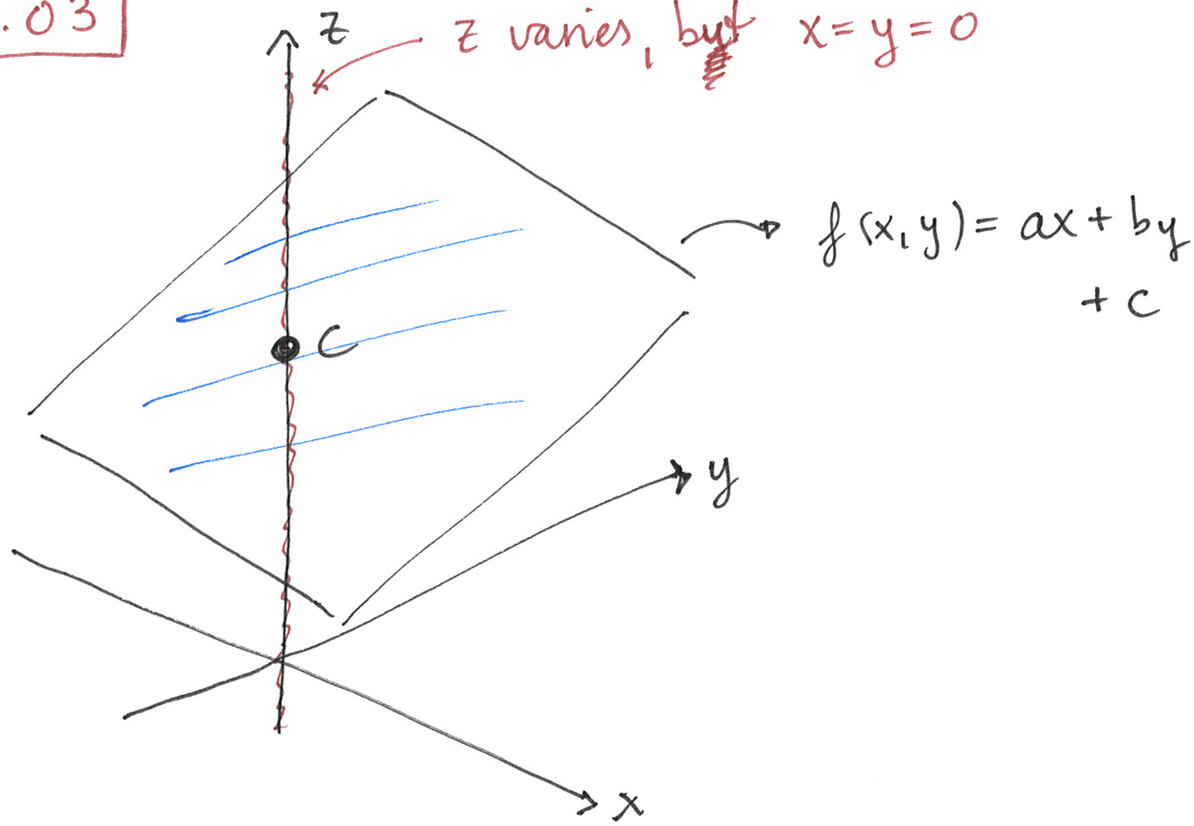
Flat sheet floating 2 above xy-coordinate system

NB: The intersection of the graph with the z-axis is $z = c$.

of $f(x,y) = ax + by + c$

$x=y=0 \Rightarrow f(0,0) = a\cdot 0 + b\cdot 0 + c = c$

START 11.03



Linear functions with c = 0

$$f(x,y) = ax + by$$

$$z = ax + by$$

$$0 = ax + by - z \iff \begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \iff$$

$$\begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \perp \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence, the graph of $f(x,y) = ax + by$:

All vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that are normal (90° angle) to $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$.

These form a plane and $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ is its normal vector.

Ex: $f(x,y) = x - 2y$

$z = x - 2y$

$0 = x - 2y - z$

$\Rightarrow$ The plane which is the graph of $f(x,y)$ has normal vector $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

Conclusion: The graph of a linear function in two variables $f(x,y) = ax + by + c$ is a plane with normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ and intersection with the z-axis $z = c$.

Partial derivatives of functions in two variables

Ex: $f(x,y) = 3x + 4y - 5$

$f(x,y) = x^2 + y^2$

Partial derivatives:

$$\underbrace{f'_x(x,y)}_{\text{"Partial derivative of } f \text{ wrt. } x\text{"}} \underbrace{:=}_{\text{is defined}} \lim_{h \to 0} \frac{\overbrace{f(x+h,y) - f(x,y)}^{\text{step only in } x\text{-direction}}}{h}$$

TO COMPUTE: Think of y as a constant. Use normal rules for differentiation to find f'_x.

$$f'_y(x,y) := \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

COMPUTE: Think of x as a constant.

Ex: i) $f(x,y) = 3x + 4y - 5$

$f'_x(x,y) = 3 + 0 - 0 = \underline{\underline{3}}$

$f'_y(x,y) = 0 + 4 - 0 = \underline{\underline{4}}$

ii) $f(x,y) = x^2 + y^2$

$f'_x(x,y) = 2x + 0 = \underline{\underline{2x}}$

$f'_y(x,y) = 0 + 2y = \underline{\underline{2y}}$

Double derivatives

ii)

$f''_{xx}(x,y) = 2$, $f''_{yy}(\cancel{x},y) = 2$

Cross derivatives

$f''_{\underbrace{xy}}(x,y) = 0$

First x, then y

$f''_{\underbrace{yx}}(x,y) = 0$

First y, then x

SAME

$\boxed{\begin{array}{l}\textbf{Def (stationary point):} \text{ Let } f(x,y) \text{ be a function.} \\ \text{A point } (x,y) = (a,b) \text{ is a } \underline{\text{stationary point}} \text{ for} \\ f \text{ if:} \\ \qquad f'_x(a,b) = 0 = f'_y(a,b)\end{array}}$

To find stationary points: Solve the system of equations:

solve for (x,y) $\left\{\begin{array}{l} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{array}\right.$