Inner product of vectors (dot product)

EBA 1180
Sect. 38
S26

Def (inner product): Let $\vec{v}$, $\vec{w}$ be n-vectors:

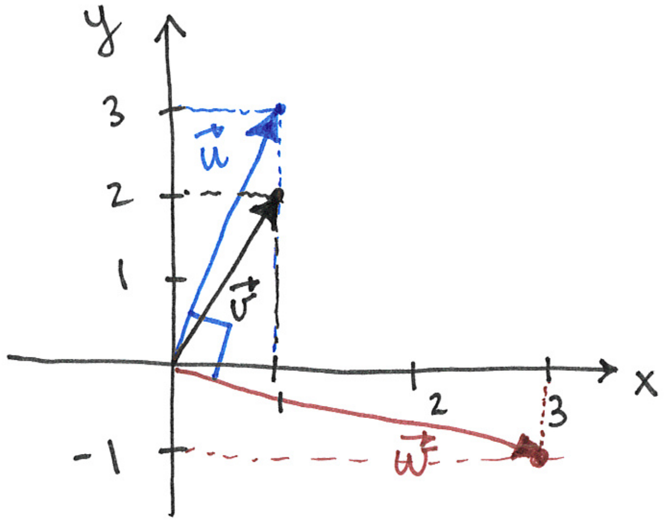
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}. \quad \text{Then,}$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{v} \cdot \vec{w} = 1 \cdot 3 + 2 \cdot (-1) = 3 - 2 = \underline{1}$$

$$\vec{u} \cdot \vec{w} = 1 \cdot 3 + 3 \cdot (-1) = 3 - 3 = \underline{\underline{0}}$$



NOTE: $\vec{u} \cdot \vec{w} = 0$

and $\vec{u} \perp \vec{w}$

"Forms a 90° angle with"

"$\vec{u}$ and $\vec{w}$ are orthogonal"

"$\vec{u}$ is normal to $\vec{w}$"

Def (orthogonal): We say that $\vec{v}$ and $\vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$

$$\boxed{\textbf{Result:}\quad \vec{v} \perp \vec{w} \iff \vec{v}\cdot\vec{w} = 0}$$

Rules of computation:

1) $\vec{v}\cdot\vec{w}$ = a number

2) $\vec{v}\cdot\vec{v} = v_1^2 + v_2^2 + \dots + v_n^2 = \|v\|^2 \geqslant 0$

$$\|\vec{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2 + \dots + v_n^2}\right)^2 = v_1^2 + v_2^2 + \dots + v_n^2$$

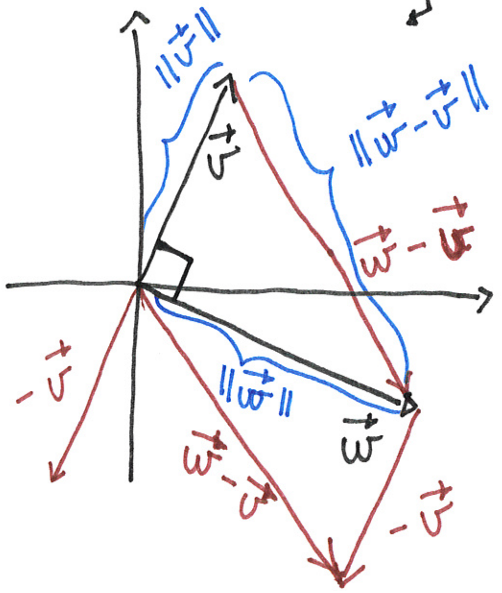
3) $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$

4) $(a\vec{u} + b\vec{w})\cdot\vec{v} = a\,\vec{u}\cdot\vec{v} + b\,\vec{w}\cdot\vec{v}$

Proof of result:

Pythagorean thm.

$$\vec{v} \perp \vec{w} \iff \|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{w} - \vec{v}\|^2$$



$\Updownarrow$ Rule 2

$$\underbrace{\left(\cancel{v_1^2} + \cancel{v_2^2} + \dots + \cancel{v_n^2}\right)}_{\|\vec{v}\|^2} + \underbrace{\left(\cancel{w_1^2} + \cancel{w_2^2} + \dots + \cancel{w_n^2}\right)}_{\|w\|^2}$$

$$= \underbrace{(w_1 - v_1)^2 + (w_2 - v_2)^2 + \dots + (w_n - v_n)^2}_{\|\vec{w} - \vec{v}\|^2}$$

$$= \cancel{w_1^2} - 2w_1v_1 + \cancel{v_1^2} + \cancel{\dots} + \cancel{w_n^2} - 2w_nv_n + \cancel{v_n^2}$$

$$\Updownarrow$$

$$0 = -2w_1v_1 - \dots - 2w_nv_n$$

$$\Updownarrow$$

$$w_1v_1 + \dots + w_nv_n = 0$$

$$\Updownarrow$$

$$\vec{w} \cdot \vec{v} = 0$$

$$\Updownarrow \text{ Rule 3}$$

$$\vec{v} \cdot \vec{w}$$

$\blacksquare$

NOTE: $\underbrace{\vec{v} \cdot \vec{w}}_{\text{inner product of n-vectors}} = \underbrace{\vec{v}^{T} \vec{w}}_{\text{matrix multiplication}}$

Ex: $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$\vec{w} \cdot \vec{v} = 2 \cdot 1 + 1 \cdot (-3) = 2 - 3 = \underline{-1}$$

SAME

$$\vec{w}^{T} \vec{v} = \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 \cdot 2 + (-3) \cdot 1 = \underline{-1}$$

$$1 \times \cancel{2} \cdot \cancel{2} \times 1$$

START 11.03

NB: $\underbrace{\vec{w}\ \vec{v}}_{\text{matrix multiplication}}$ is not defined $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$2 \times 1 \cdot 2 \times 1$$

Not the same

Functions in two variables

Ex: $f(x,y) = 2x + 3y - 1$, linear function

two variables

$f(x,y) = x^2 + y^2$, polynomial function

$f(x,y) = \dfrac{x+y}{x-y}$, rational function

$f(x,y) = xe^y$

General: $f(x,y)$: function expression in x, y

x, y : input variables

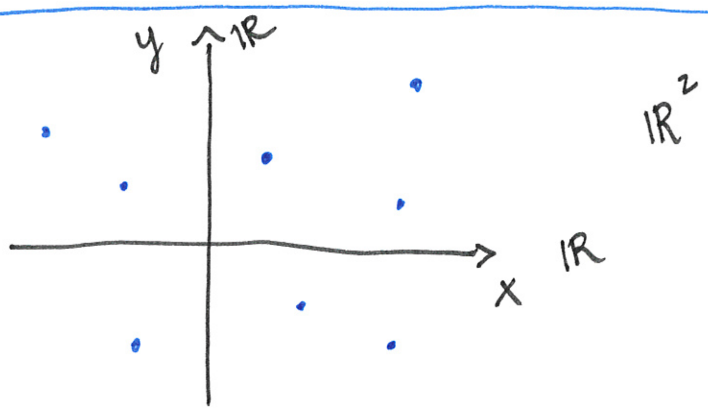
$z = f(x,y)$: output variable

Def (Domain of f):

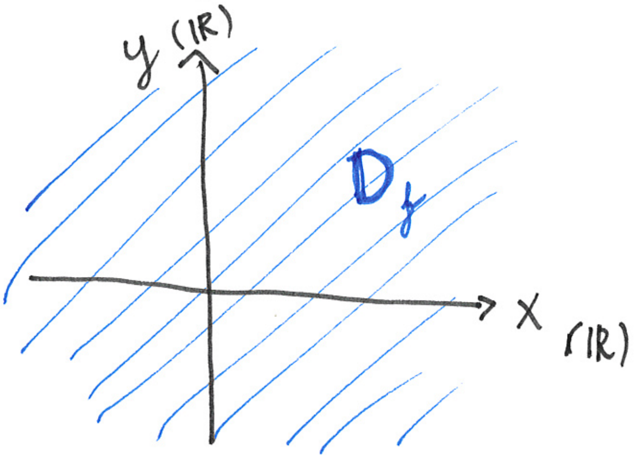
D_f = domain of f = all coordinate pairs (x,y) that we can input into our function f

Subset of the xy-plane, $\mathbb{R}^2$

"R two"



Ex: $f(x,y) = 2x + 3y - 1$, $D_f = \mathbb{R}^2$ (any real value for x, any real value for y)



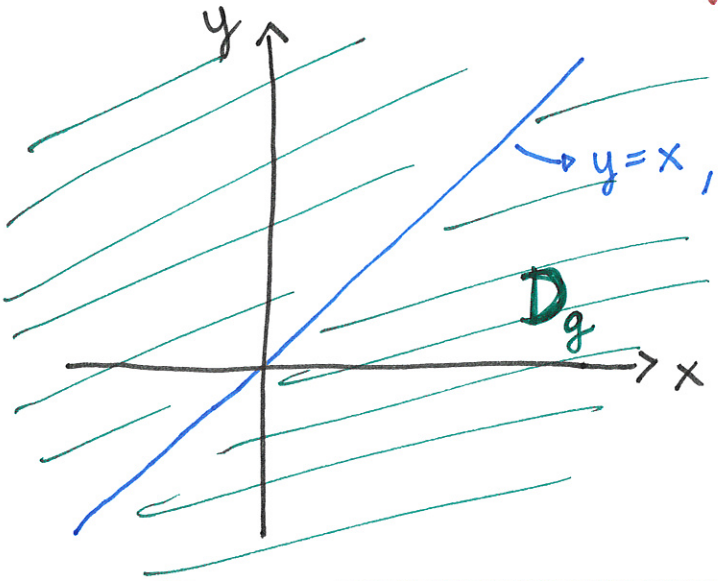
$$g(x,y) = \frac{x+y}{x-y},$$

D_g : $x \neq y$ → Divide by 0 if $x=y$

$$\underbrace{D_g}_{\text{domain of } g} = \underbrace{\{}_{\text{the set of}} (x,y) \underbrace{\in}_{\text{element in}} \mathbb{R}^2 \underbrace{:}_{\text{such that}} x \neq y \}$$

← (set of)

$\boxed{\text{Natural domain}}$



→ $y=x$, NOT in D_g

Def (range): V_f = range of f = all values $f(x,y)$ can attain when $(x,y) \in D_f$

To find the range (of a continuous function):

Find the max/min of f.

Ex:

$$f(x,y) = 2x + 3y - 1$$

From ex: $D_f = \mathbb{R}^2$

$$V_f = (-\infty, \infty) = \underline{\underline{\mathbb{R}}}$$

$y=0$, $x \to \infty$:

$f(x,y) \to \infty$

$y=0$, $x \to -\infty$:

$f(x) \longrightarrow -\infty$

$$g(x,y) = x^2 + y^2$$

$$D_g = \mathbb{R}^2$$

$$V_g = [0, \infty)$$

Can only get non-neg. numbers because of squares

Graph and level curves

Def (graph of function in two variables):

The graph of a function f in two variables is the set of all points

$$(x,y,z)$$

where $(x,y) \in D_f$ and $z = f(x,y)$.

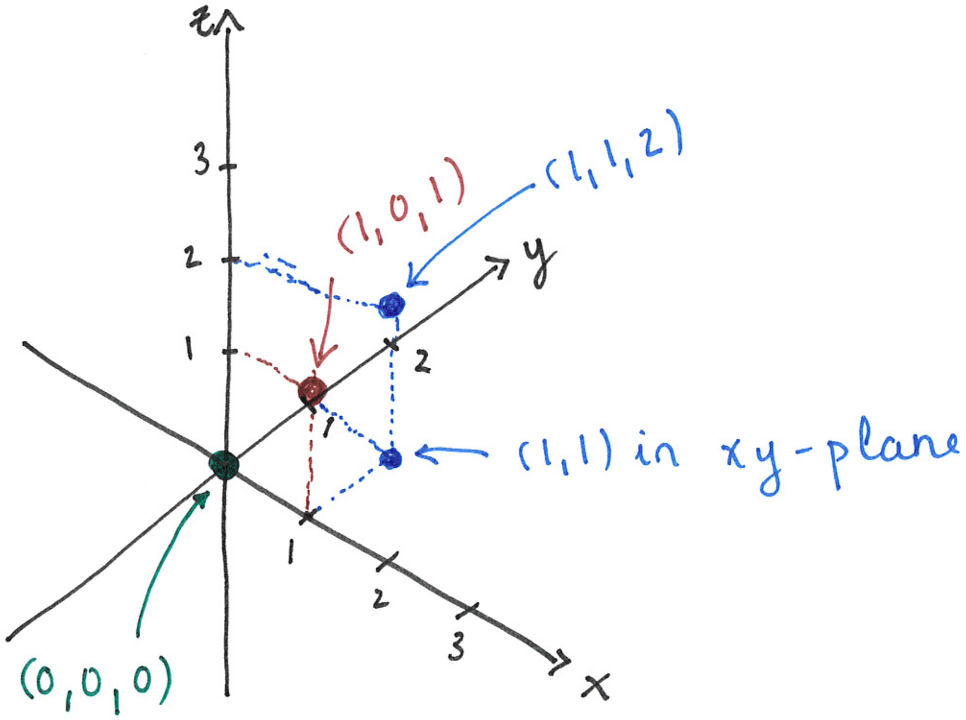
Can draw the graph of f in an xyz-coordinate system.

$\mathbb{R}^3$: 3d space

<u>Ex</u>: $f(x,y) = x^2 + y^2$, $D_f = \mathbb{R}^2$

<u>Make a table and plot points:</u>

(x,y)	$(0,0)$	$(1,0)$	$(1,1)$
$z = f(x,y)$	$0^2+0^2=0$	$1^2+0^2=1$	$1^2+1^2=2$
Point in xyz-plane	$(0,0,0)$	$(1,0,1)$	$(1,1,2)$



The graph of f is called a <u>surface</u>.