Inverse matrices

EBA1180
S26
Lect. 37

Ex:
$$2x + y = 4$$
$$x + 2y = 3$$

Matrix form:
$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$
$$A \vec{x} = \vec{b}$$

If A^{-1} exists:
$$\underbrace{A^{-1}A}_{I} \vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

Last lecture:
$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Advantage of A^{-1}:
Quickly solve lin. syst. with many different r.h.s.

$$\vec{x} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8-3 \\ -4+6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \end{bmatrix}$$

So: $(x, y) = \underline{\underline{\left(\frac{5}{3}, \frac{2}{3}\right)}}$

Computing inverse matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$|A| = 1\,(\underbrace{18-12}_{6}) - 1\,(\underbrace{9-4}_{5}) + 1\,(\underbrace{3-2}_{1})$$

$= 2 \neq 0$, so A has an inverse.

FORMULA TO FIND A^{-1}:

Adjoint matrix of A: $\operatorname{adj}(A)$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T$$

Compute c_{ij}'s: $c_{11} = 6$ $\quad c_{12} = -5$ $\quad c_{13} = 1$

$c_{21} = -6$ $\quad c_{22} = 8$ $\quad c_{23} = -2$

$c_{31} = 2$ $\quad c_{32} = -3$ $\quad c_{33} = 1$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}^T$$

$$= \frac{1}{2} \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Alternative way of finding A^{-1}:

A: $n \times n$ matrix

$$[A \mid \underbrace{I}_{\text{same size as } A}] \sim \cdots \sim [B \mid C]$$

Reduced echelon form:

Aim for $B = I$

2 cases:

$B = I$: $A^{-1} = C$

$B \neq I$: A^{-1} doesn't exist

Alternative method:

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, seen before

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \begin{matrix} \\ -2 \end{matrix}$$

$\underbrace{\quad}_{A}$ $\underbrace{\quad}_{I}$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{array}\right] :(-3) \sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \begin{matrix} \\ -2 \end{matrix}$$

Echelon form

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right] = \left[I \mid A^{-1} \right]$$

Reduced echelon form

So:

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Why does this work? 3×3

$$A = \begin{bmatrix} 1 & \cdots & \\ 2 & \cdots & \\ \cdots & & \end{bmatrix} \xrightarrow{-2} \sim \begin{bmatrix} 1 & \cdots & \\ 0 & \cdots & \\ \cdots & & \end{bmatrix} = B, \text{ then}$$

1st elementary row op.

$$B = \underbrace{E_1}_{\text{elementary matrix}} \cdot A$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

the matrix found by doing the elementary row operation on the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2} \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

Each row operation corresponds to matrix multiplication

START 11.03

$$[A \mid I] \underbrace{\sim}_{\text{row op. } 1} \cdots \underbrace{\sim}_{\text{row op. } r} [B \mid C]$$

Aim for I

$$\Updownarrow$$

$$\left.\begin{aligned} \underbrace{E_r}_{\text{row op. } r} \cdots \underbrace{E_2}_{\text{row op. } 2} \underbrace{E_1}_{\text{row.op. } 1} A &= B \\ E_r \cdots E_2 E_1 I &= C \end{aligned}\right\}$$

Multiply with one elementary matrix for each row operation.

Then, if $B = I$:

$$\left.\begin{aligned} \overbrace{E_r \cdots E_2 E_1}^{C} A &= I \\ \underbrace{E_r \cdots E_2 E_1} &= C \end{aligned}\right\}$$

So, $\underbrace{C}_{A^{-1}} A = I$

If $B \neq I$: $|A| = 0 \Rightarrow$ No inverse.

Will have 0 in diagonal $\Rightarrow |B| = 0$. But:

$|A| = \overbrace{|B|}^{0} \cdot \text{scaling} \cdot \text{switch sign} \Rightarrow |A| = 0$

Exercise set 36

7) b), c), d)

Profit per stock

a)

$$\underbrace{\overset{\text{purchase price asset 1}}{60}\,\underset{\text{\# of units of 1st risky asset}}{x} + \overset{\text{purchase price asset 2}}{75}\,\underset{\text{\# of units of 2nd risky asset}}{y} + \overset{\text{purchase price asset 3}}{320}\,\underset{\text{\# of units of 3rd risky asset}}{z}}_{\text{total investment in risky assets}} = \underbrace{400\,000}_{\text{all money we have}}$$

total investment in risky assets = all of our money

$\boxed{\text{Hence, a budget constraint.}}$

b), c), d) : Profit per stock

	A ^{x}	B ^{y}	C ^{z}
1	20	5	30
2	40	-50	180
3	-20	25	-265

Profit of owning 1 unit of stocks A, B, C resp. in scenario 3

Profit of owning 1 unit of stock A in each of the scenarios

Budget: $60x + 75y + 320z = C$ $\to$ $\boxed{400\ 000}$

WHICH PROFITS ARE POSSIBLE?

Profits (R_1, R_2, R_3)

Profit in scenario 1: $20x + 5y + 30z = R_1$

— " — 2: $40x - 50y + 180z = R_2$

— " — 3: $-20x + 25y - 265z = R_3$

+ budget constraint.

Gaussian elimination:

$$\left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 40 & -50 & 180 & R_2 \\ -20 & 25 & -265 & R_3 \\ 60 & 75 & 320 & C \end{array}\right] \quad -2,\ 1,\ -3$$

$$\sim \left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 30 & -235 & R_3 + R_1 \\ 0 & 60 & 230 & C - 3R_1 \end{array}\right] \quad \frac{1}{2},\ 1$$

$$\sim \left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 0 & -175 & R_3 + R_1 + \frac{1}{2}(R_2 - 2R_1) \\ 0 & 0 & 350 & C - 3R_1 + (R_2 - 2R_1) \end{array}\right] \quad 2$$

$$\sim \left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 0 & -175 & R_3 + \frac{1}{2}R_2 \\ 0 & 0 & 0 & \boxed{C - 5R_1 + 2R_2 + 2R_3} \end{array}\right]$$

For system to have a solution, must have:

$$C - 5R_1 + 2R_2 + 2R_3 = 0$$