

Geometric interpretation of vectors

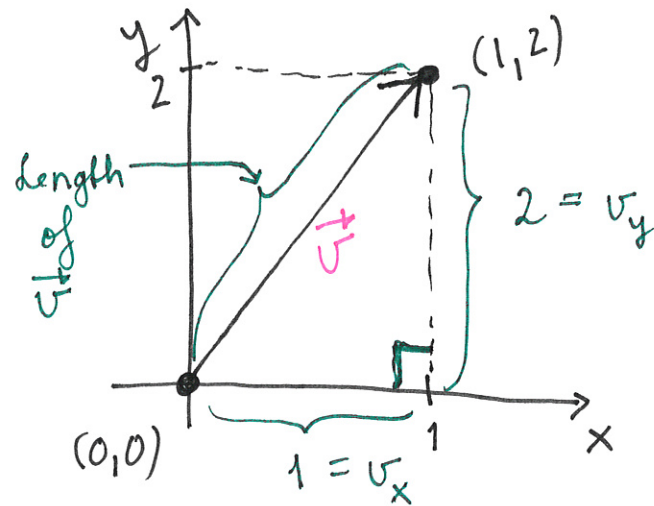
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Lect. 9

S.26

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

corresponds to an arrow from $(0,0)$ to $(v_x, v_y) = (1,2)$.



Pythagorean theorem:

$$v_x^2 + v_y^2 = (\text{length of } \vec{v})^2$$

$$\text{length of } \vec{v} = \sqrt{v_x^2 + v_y^2}$$

A vector has length and direction.

$$\|\vec{v}\| = \left\| \begin{bmatrix} v_x \\ v_y \end{bmatrix} \right\| = \sqrt{v_x^2 + v_y^2}$$

Q: $\|\vec{v}\| = \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2} = \underline{\underline{\sqrt{5}}}$

Def (Length of vector): If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

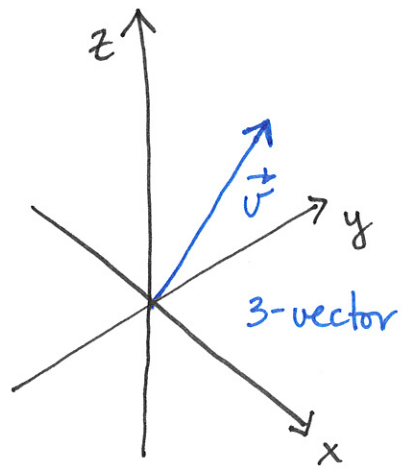
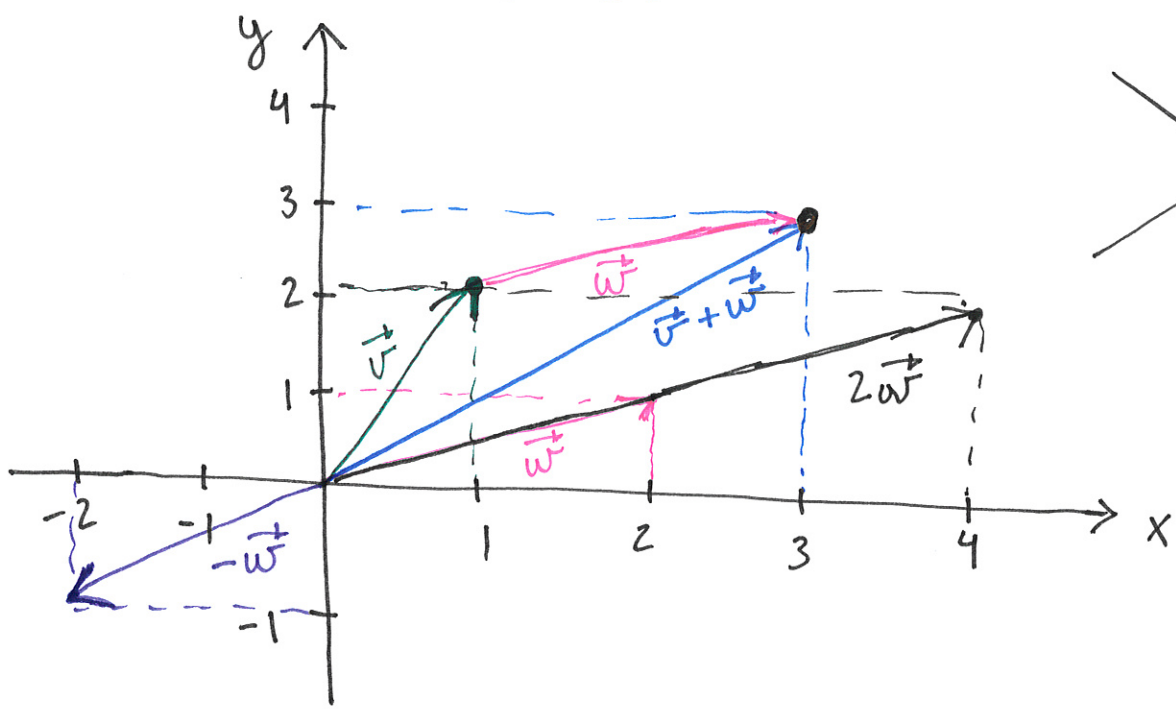
Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

ADD: $\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

SCALAR MULTIPLICATION: $2\vec{w} = 2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$(-1)\vec{w} = (-1) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

VISUALIZE VECTOR OPERATIONS:



Determinants "the determinant of A"

A
n x n matrix
SQUARE!

$\det(A) = |A|$
a number

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\det(A) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

2×2

$= 2 \cdot 2 - 1 \cdot 1 = \underline{\underline{3}}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$

FORMULA (Determinant, $n=2$)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

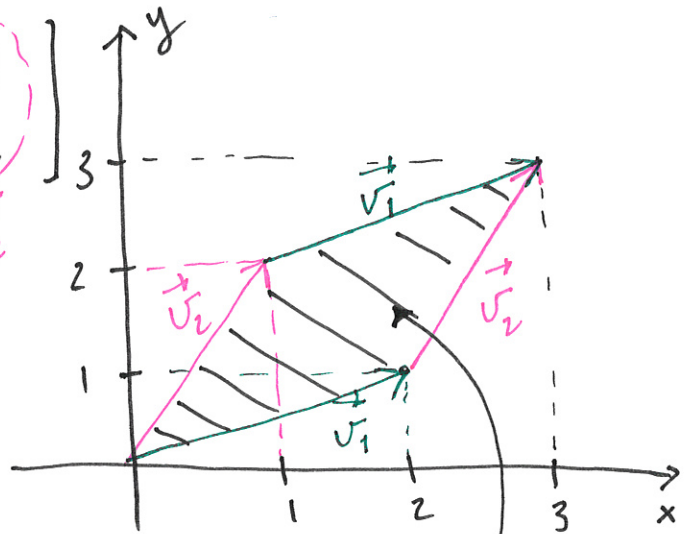
Interpretation:

$A = \begin{bmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

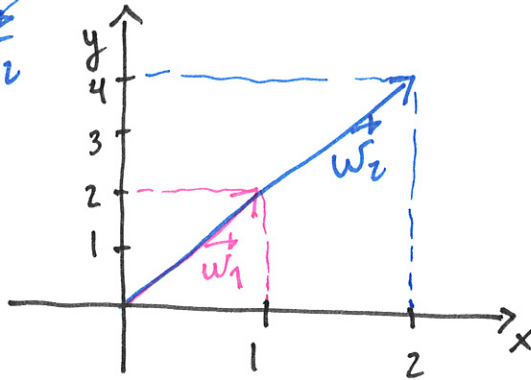
Can prove: $|\det(A)| =$ area of parallelogram spanned by \vec{v}_1 and \vec{v}_2 .



NOTE: $\begin{vmatrix} \vec{v}_2 \\ \vec{v}_1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 \cdot 1 - 2 \cdot 2 = -3$

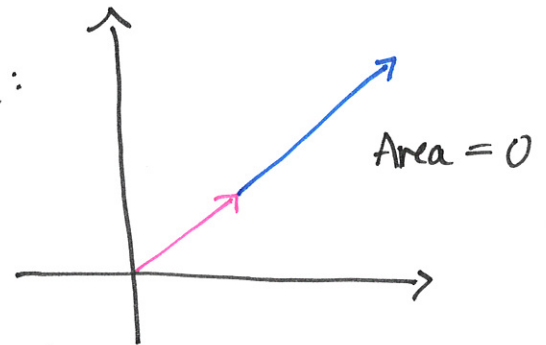
EX: $\begin{vmatrix} \vec{w}_1 \\ \vec{w}_2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 2 = 0$

Area can't be negative: must take absolute value



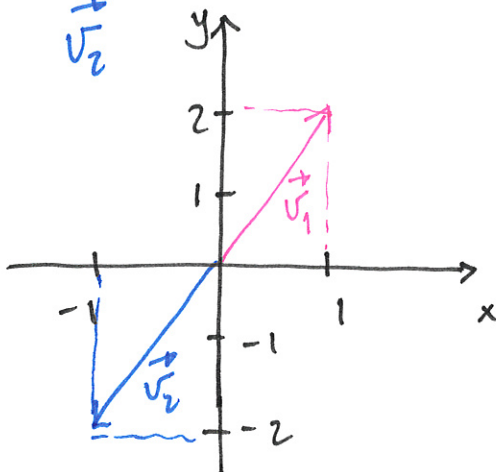
Parallelogram spanned by \vec{w}_1 and \vec{w}_2 :

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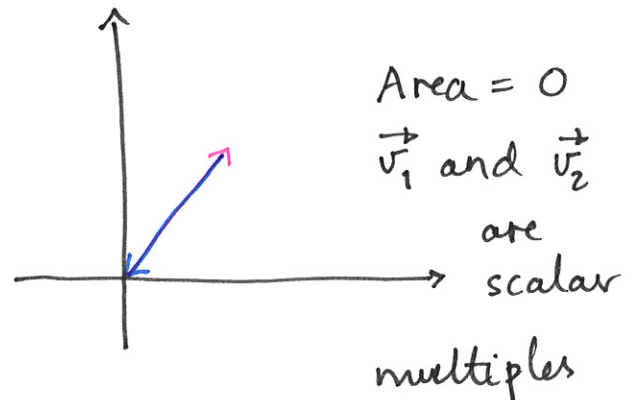


\vec{w}_1 and \vec{w}_2 are scalar multiples.

EX: $\begin{vmatrix} \vec{v}_1 \\ \vec{v}_2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = 1 \cdot (-2) - (-1) \cdot 2 = -2 + 2 = 0$



Parallelogram spanned by \vec{v}_1 and \vec{v}_2 :



Result: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \iff \vec{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ are scalar multiples

The general case, $n \times n$: METHOD for finding determinants

Cofactor expansion

A , $n \times n$ matrix

Ex:

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$, 3×3 square

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot c_{11} + 1 \cdot c_{12} + 1 \cdot c_{13}$

Cofactor expansion along row 1

where c_{11}, c_{12}, c_{13} are cofactors:

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

minor

$c_{ij} := (-1)^{i+j} M_{ij}$

where M_{ij} is the determinant of the submatrix you get when you delete row i , column j .

So:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 1 \cdot (-1) \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 18 - 12 - (9 - 4) + 3 - 2 = \underline{\underline{2}}$$

NB: • Cofactor expansion along any row/column gives same result.

- Any determinant can be computed via cofactor expansion.
- Exploit the 0's! Use row/column with lots of zeros to simplify the calculations.

Connection between # solutions of linear systems and determinants



- Start with $n \times n$ linear system (# equations = # variables). Then, the corresponding coefficient matrix is an $n \times n$ matrix.

Ex: $x + y + z = 3$
 $x + 2y + 4z = 7$
 $x + 3y + 9z = 13$

3×3 linear system

Write as: $A \vec{x} = \vec{b}$ (matrix form)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix}$$

coefficient matrix

Result: i) $|A| \neq 0 \Rightarrow$ One unique solution

$A_{n \times n}$

ii) $|A| = 0 \Rightarrow$ No solutions OR infinitely many solutions

Theorem (Cramer's rule):

Consider a linear system, $A \vec{x} = \vec{b}$, with coefficient matrix A and r.h.s. \vec{b} , such that A is square ($n \times n$) with $|A| \neq 0$. Then, the solution of the linear system

is:

$$x_1 = \frac{|A_1(\vec{b})|}{|A|}, x_2 = \frac{|A_2(\vec{b})|}{|A|}, \dots,$$

$$x_n = \frac{|A_n(\vec{b})|}{|A|}$$

where $A_i(\vec{b})$ is the matrix you get when you replace the i 'th column of A with \vec{b} .