More linear systems: Number of solutions

EBA 1180
Lect. 8
S. 26

Ex: $x_1 + x_2 + x_3 + x_4 + x_5 = 17$

$x_1 - 2x_2 - x_3 + 4x_5 = 8$

$2x_1 + x_2 - 5x_3 + 7x_4 = 11$

Def (Pivot position): A pivot position is a position where there is a pivot in the echelon form.

Ex ctd.:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 1 & -2 & -1 & 0 & 4 & 8 \\ 2 & 1 & -5 & 7 & 0 & 11 \end{array}\right] \begin{array}{l} \\ -1 \\ -2 \end{array}$$

$\sim$

Add $(-1)*$ row 1 to row 2

Add $(-2)*$ row 1 to row 3

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -3 & -2 & -1 & 3 & -9 \\ 0 & -1 & -7 & 5 & -2 & -23 \end{array}\right]$$

$\sim$

switch rows 2 and 3

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & -7 & 5 & -2 & -23 \\ 0 & -3 & -2 & -1 & 3 & -9 \end{array}\right] \begin{array}{l} \\ \\ -3 \end{array}$$

add $(-3)*$row 2 to row 3

$$
\sim \quad
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left[\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 17 \\
0 & -1 & -7 & 5 & -2 & -23 \\
0 & 0 & 19 & -16 & 9 & 60
\end{array}\right]
\end{array}
$$

Pivots!

Echelon form

Pivot positions: $(1,1)$, $(2,2)$, $(3,3)$.

The linear system has two degrees of freedom: x_4, x_5 are free. Hence, the system has infinitely many solutions.

Why? From the echelon form:

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 17 \\
-x_2 - 7x_3 + 5x_4 - 2x_5 &= -23 \\
19x_3 - 16x_4 + 9x_5 &= 60 \Rightarrow 19x_3 = 60 + 16x_4 - 9x_5
\end{aligned}
$$

$\vdots$

$x_2 = \ldots$ via x_4 and $x_5 \ldots$

$\vdots$

$x_1 = \ldots$ via x_4 and $x_5 \ldots$

$\leadsto$ Can choose any x_4 and x_5 and the original linear system still holds.

<u>Result</u>: For any linear system, the pivot positions in echelon form, determine the number of solutions.

<u>Different cases</u>:

in extended matrix form

i) <u>Pivot position in the last column</u>: <u>No</u> solutions.

<u>Ex</u>:

$$\left[\begin{array}{ccc|c} \cdot & \cdots & \cdot & \cdot \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{array}\right] \to \text{Says:}$$

$0 = 1$; Not true (never)

ii) <u>No pivot position in the last column</u>:

The linear system has solutions.

a) <u>Pivot positions in all variable columns</u>:

<u>One</u> solution.

<u>Ex</u>:

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & \cdots & \cdots & \vdots \\ 0 & 1 & \cdots & \vdots \\ 0 & 0 & 1 & \vdots \end{array}\right] \end{array} \begin{array}{l} \to x_1 = \text{number} \\ \to x_2 = \text{number} \\ \to x_3 = \text{number} \end{array}$$

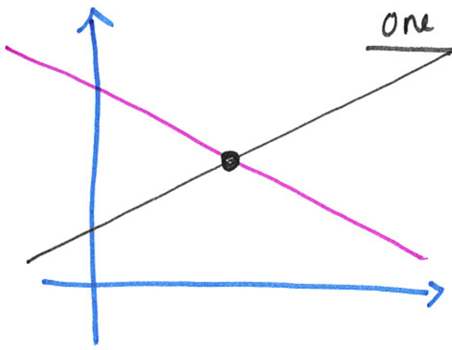
b) There are variable columns without pivot positions:

Infinitely many solutions.

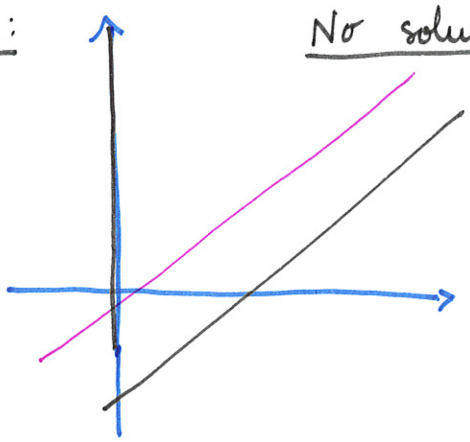
Ex:

$$\left[\begin{array}{cccccc|c} 1 & \cdots & & & & & \cdots \\ 0 & 1 & \cdots & & & & \cdots \\ 0 & 0 & 1 & \cdots & & & \cdots \end{array}\right]$$

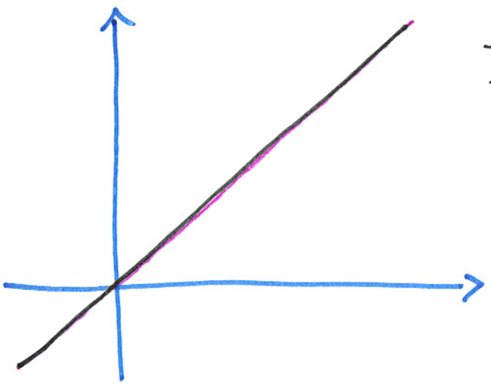
one solution:



No solutions:



Infinitely many solutions:



Theorem: Any linear system has either:

i) No solutions. $\rightsquigarrow$ Inconsistent

ii) One unique solution.
iii) Infinitely many solutions.
$\Big\}$ Consistent

START 11.02

Computations with matrices and vectors

Def ($m \times n$ matrix):

" m times n "
" m by n "

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Ex:

$$A = \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 7 & -1 & 0 \end{bmatrix}}_{3 \text{ columns}} \Big\} \ 2 \text{ rows}$$

Capital letters for matrices

Dimension: 2×3 matrix

$$A = \begin{array}{ccc|c} & & & \\ \end{array}$$

$$A = \begin{array}{cccc} \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] & \begin{array}{c} 1 \\ 2 \end{array} \\ \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} & \end{array}$$

a_{12}: row 1, column 2

- **Addition:** $A + B$
- **Subtraction:** $A - B$

Defined if A and B have the same dimension (e.g. both $m \times n$)

Result is a matrix of same dimension as A (B)

- Scalar multiplication: $r \cdot A$ → Result will be a matrix of the same size as A

r: scalar (number)
A: matrix

Always defined!

Ex:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2+(-1) & 3+1 \\ -1+1 & 0+2 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

Do addition/subtraction position by position.

Ex:

$$2 \cdot \begin{bmatrix} 1 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 4 \\ 2\cdot(-1) & 2\cdot 2 \\ 2\cdot 0 & 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -2 & 4 \\ 0 & 2 \end{bmatrix}$$

Do multiplication by scalar position by position.

Def (n-vector): An n-vector is a matrix with n rows and 1 column (a column vector).

- Write vectors as $\vec{v} = \mathbf{v} = \underline{v}$

lower case letters for vectors

$\underbrace{\mathbf{v}}_{\text{bold face } v}$

Ex:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$; a 3-vector viewed as a column vector.

$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ Row vector

Vector operations

- Addition: $\vec{v} + \vec{w}$
- Subtraction: $\vec{v} - \vec{w}$

Defined for vectors of the same length

- Scalar multiplication: $\underbrace{r \cdot \vec{v}}$ (r scalar/number)

Always defined

TASK:

ADD: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2+(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}}$

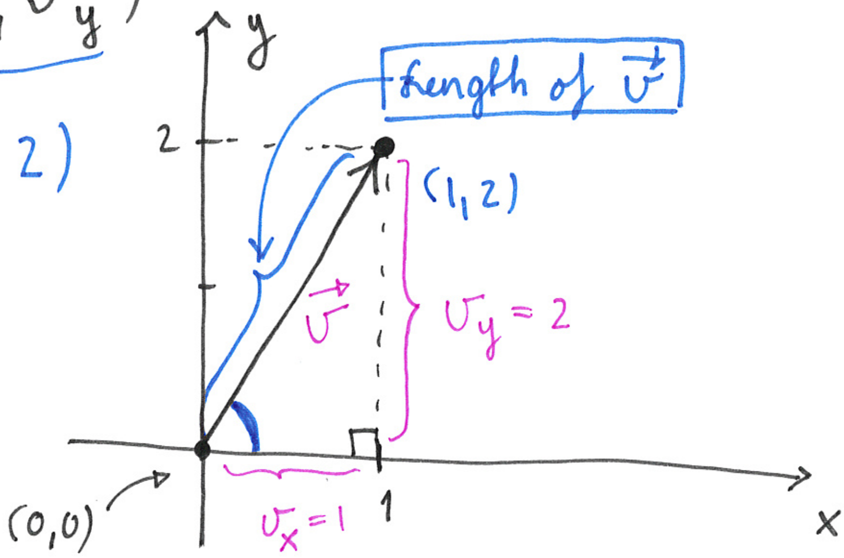
SUBTRACT: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 2-(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}}$

SCALAR MULTIPLICATION: $2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}}$

Geometric interpretation of vectors

Ex: $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}v_x\\v_y\end{bmatrix}$ → Equality for matrices (vectors) means equality element by element

Corresponds to an arrow from $(0,0)$ to  $\underbrace{(v_x, v_y)}_{} = (1, 2)$



A vector has length (magnitude) and a direction.