New theme:

Systems of equations

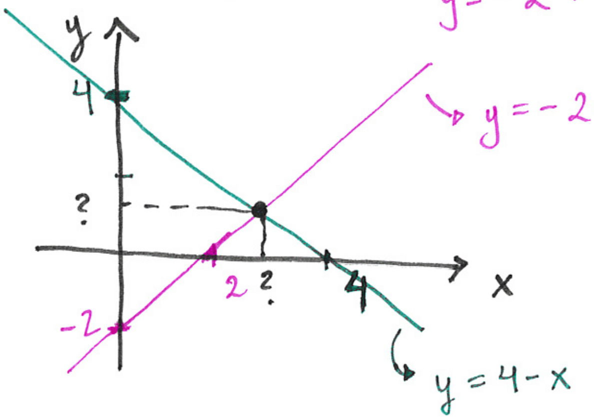
EBA1180
Lecture 7
S26

Some types of systems of equations :

LINEAR

i) $x + y = 4 \rightarrow y = 4 - x$

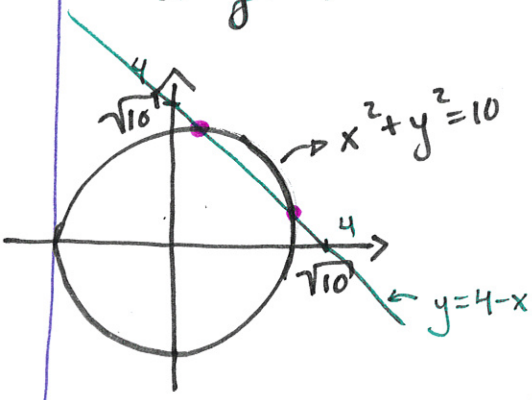
$x - y = 2 \rightarrow y = -2 + x$



NON-LINEAR

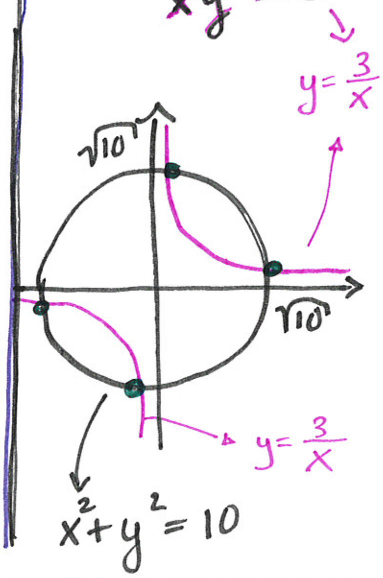
ii) $x^2 + y^2 = 10$

$x + y = 4$



iii) $x^2 + y^2 = 10$

$xy = 3 \rightarrow y = \frac{3}{x}$



SOLVE?

i) 2 methods

Eliminate:

$$x + y = 4$$
$$+(x - y = 2)$$
$$2x = 6$$
$$\underline{x = 3}$$
$$y = x - 2 = 3 - 2 = \underline{1}$$

$(x, y) = (3, 1)$

Substitute:

$$x + y = 4 \Rightarrow y = 4 - x$$
$$x - y = 2$$
$$x - (4 - x) = 2$$
$$2x - 4 = 2$$
$$2x = 6$$
$$\underline{x = 3}$$
$$y = 4 - x = 4 - 3 = \underline{1}$$

SAME $\rightarrow (x, y) = (3, 1)$

ii) $x + y = 4$

$$y = 4 - x$$
$$x^2 + (4 - x)^2 = 10$$
$$x^2 + 16 - 8x + x^2 = 10$$
$$2x^2 - 8x + 6 = 0 \quad | :2$$
$$x^2 - 4x + 3 = 0$$
$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 3}}{2 \cdot 1}$$
$$= \frac{4 \pm \sqrt{4}^{\rightarrow 2}}{2} \Rightarrow \underline{x_1 = 3}, \underline{x_2 = 1}$$

$y_1 = 4 - x_1 = 4 - 3 = \underline{1}$ and

$y_2 = 4 - x_2 = 4 - 1 = \underline{3}$

ii) (ctd.) So: $(x_1, y_1) = (3,1)$ and $(x_2, y_2) = (1,3)$

iii) $xy = 3$

$y = \frac{3}{x}$ ($x \neq 0$ since $xy = 3$)

$$x^2 + \left(\frac{3}{x}\right)^2 = 10$$

$$x^2 + \frac{9}{x^2} = 10$$

$$x^4 + 9 = 10x^2$$

$$x^4 - 10x^2 + 9 = 0 \quad \longrightarrow \quad x^2 = u$$

$$(\underbrace{x^2}_{u})^2 - 10\underbrace{x^2}_{u} + 9 = 0$$

$$u^2 - 10u + 9 = 0 \quad \longrightarrow \quad u = \frac{10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot 9}}{2 \cdot 1} \;\vdots\; u_1 = 9, \; u_2 = 1$$

abc-formula:

$$x_1^2 = 9, \quad x_2^2 = 1$$

$$x_1 = \pm 3, \quad x_2 = \pm 1 \quad \longrightarrow \quad \boxed{y = \frac{3}{x}}$$

$$\{(x,y)\} = \{(3,1), (-3,-1), (1,3), (-1,-3)\}$$

"The set of"

Def (Linear system): An $m \times n$ linear system in the variables $x_1, x_2, \ldots, x_n$ is a system of m linear equations in $x_1, \ldots, x_n$. It has the form:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \quad \vdots \quad &\quad \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n}_{n \text{ unknowns}} &= b_m \end{aligned}\right\} \begin{matrix} m \\ \text{eqns.} \end{matrix}$$

where $a_{11}, a_{21}, \dots, a_{mn}$ and $b_1, b_2, \dots, b_m$ are given numbers.

<u>Ex</u>:

$$3 \text{ eqns} \left\{ \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 7 \\ x_1 - x_2 \qquad + 2x_4 &= 10 \\ \underbrace{x_1 + x_2 - x_3 \qquad}_{4 \text{ unknowns}} &= 3 \end{aligned} \right.$$

$\rightsquigarrow$ 3×4 linear system.

<u>Ex</u>:

$$3 \text{ eqns} \left\{ \begin{aligned} x + y + z &= 3 \quad (1) \\ x + 2y + 4z &= 7 \quad (2) \\ \underbrace{x + 3y + 9z}_{3 \text{ unknowns/variables}} &= 13 \quad (3) \end{aligned} \right. \rightsquigarrow 3 \times 3 \text{ linear system}$$

<u>SOLVE?</u>

(1) $\boxed{x = 3 - y - z}$

(2) $3 - y - z + 2y + 4z = 7$

$y + 3z = 4 \Rightarrow \boxed{y = 4 - 3z}$

(3) $3 - y - z + 3y + 9z = 13$

$$2y + 8z = 10 \;\longrightarrow\; 2(4-3z) + 8z = 10$$

$$2z = 2$$

$$\underline{\underline{z = 1}}$$

$y = 4 - 3z = 4 - 3 \cdot 1 = \underline{\underline{1}}$

$x = 3 - y - z = 3 - 1 - 1 = \underline{\underline{1}}$

Solution: $\underline{\underline{(x, y, z) = (1, 1, 1)}}$

Gaussian elimination

General and systematic method to solve <u>any</u> linear system.

METHOD

t.b.e

1) Write down the <u>augmented matrix</u> of the system.

2) Use <u>elementary row operations</u> until you reach <u>echelon form</u>.

3) Use <u>back substitution</u> to solve the system.

<u>Ex</u>:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}$$

MATRIX

1) <u>Augmented matrix</u>:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1^{0} & 2 & 4 & 7 \\ 1^{0} & 3^{0} & 9 & 13 \end{array}\right]$$

AIM

Elementary row operations

i) Switch two rows.

ii) Multiply a row by a constant $c \neq 0$.

iii) Add a multiple (by a non-zero constant) of a row to another row.

2) Gaussian:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right]$$

row equivalent to

-1 times first row and add to second row

$[-1 \quad -1 \quad -1 \mid -3]$

$$\sim \quad \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \end{array}$$

-1 times row 1, add row 3

$$y + 3z = 4$$
$$2y + 8z = 10$$

$$\sim \quad \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right] \end{array}$$

(-2) * row 2 add to row 3

$[0 \quad -2 \quad -6 \mid -8]$

$$y + 3z = 4$$
$$2z = 2$$

ECHELON FORM!

Pivot: The first non-zero element in a row is called a pivot.

Echelon form: An echelon form is where
1) All entries below a pivot are 0.
2) If some rows are all zeros, they're at the bottom of the matrix.

3) **Gaussian**: To solve from echelon form:

Back substitution:

1) Start with last eqn.
2) Work back, substituting the variables we solved for:

Ex:

$$x + y + z = 3 \rightarrow x + 1 + 1 = 3 \Rightarrow \underline{x = 3 - 2 = 1}$$

$$y + 3z = 4 \rightarrow y + 3 \cdot 1 = 4 \Rightarrow \underline{y = 1}$$

$$2z = 2 \Rightarrow \underline{z = 1}$$

Solution: $(x, y, z) = (1, 1, 1)$