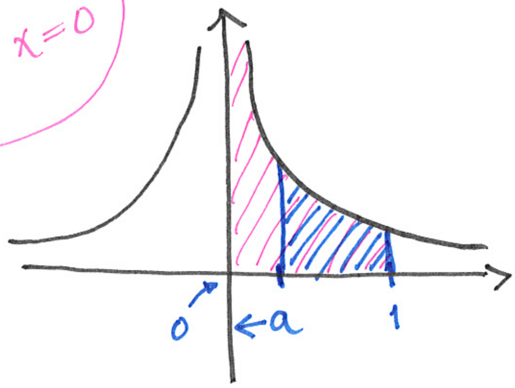
Improper integrals (continued)

EBA1180
S26
Lecture 6
(30)

Ex: $$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,dx$$

$\frac{1}{x^2}$ is not defined for $x=0$

THIS MEANS



$$\lim_{x\to 0^\pm} \frac{1}{x^2} = \infty$$

$$\lim_{x\to \pm\infty} \frac{1}{x^2} = 0$$

$$\int_a^1 \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_{x=a}^1 = \left[-\frac{1}{x}\right]_{x=a}^1$$

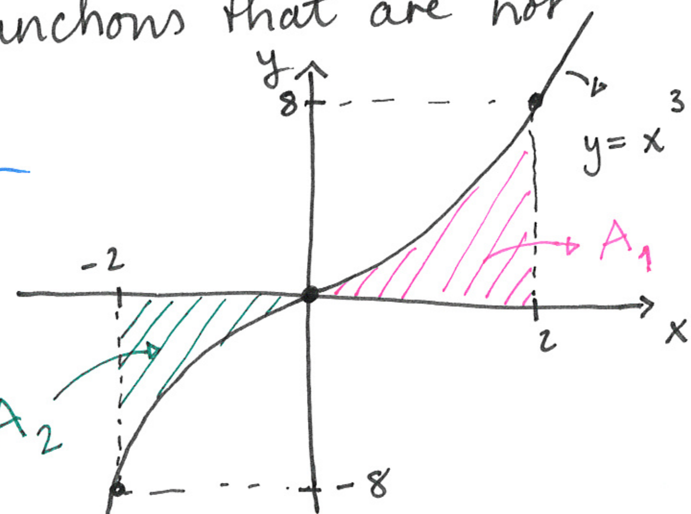
$$= \underline{-1+\frac{1}{a}}$$

Then, $$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,dx$$

$$= \lim_{a\to 0^+} \left(-1+\overset{\to\infty}{\frac{1}{a}}\right) = \underline{\underline{\infty}}$$

What about integration of functions that are not always ≥ 0 ?

Ex: $$\int_{-2}^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_{x=-2}^2$$



$$= \frac{1}{4}\left(2^4 - (-2)^4\right) = \frac{1}{4}(16 - 16) = 0 \rightarrow \text{WHY?}$$

$$I_1 = \int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_{x=0}^{2} = \frac{1}{4}\left(\overbrace{2^4}^{16} - 0^4\right)$$

$$= \underline{4}$$

$$I_2 = \int_{-2}^{0} x^3\,dx = \left[\frac{1}{4}x^4\right]_{x=-2}^{0} = \frac{1}{4}\left(\overbrace{(0)^4}^{0} - \overbrace{(-2)^4}^{16}\right)$$

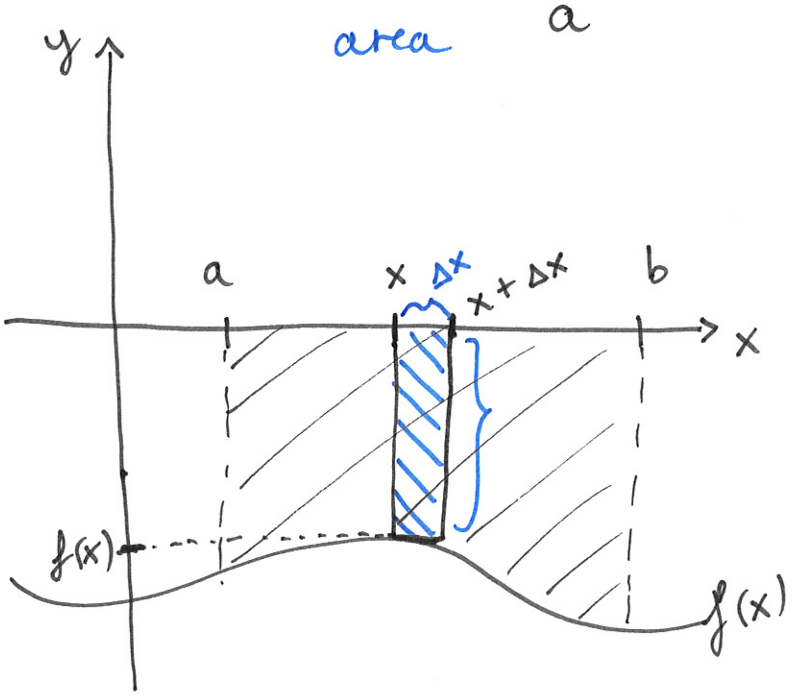
$$= \frac{-16}{4} = \underline{-4}$$

<u>So</u>: $I_1 + I_2 = 4 + (-4) = 0 = A_1 - A_2$

<u>When $f(x) \leq 0$ in $[a,b]$:</u>

Area between the x-axis and the graph of $y = f(x)$ in $[a,b]$ is:

$$\underbrace{A}_{\text{area}} = \int_a^b -f(x)\,dx, \quad \text{so} \quad \int_a^b f(x)\,dx = -A$$



<u>Height of rectangle:</u>

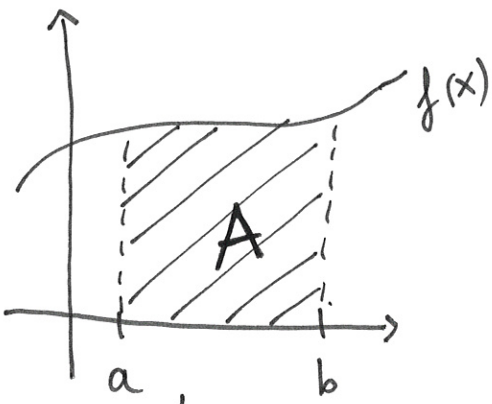
$$0 - f(x) = -f(x)$$

a positive number

<u>Area of rectangle</u> $= \underbrace{-f(x)}_{\text{height}}\,\Delta x$

3 cases

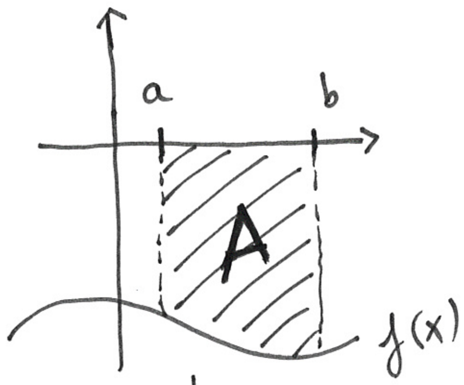
i) $f(x) \geq 0$:



$$A = \int_a^b f(x)\,dx$$

(as before)

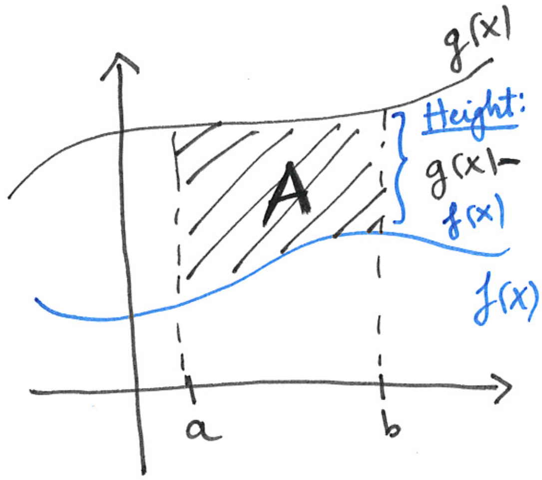
ii) $f(x) \leq 0$



$$A = \int_a^b -f(x)\,dx$$

$$-A = \int_a^b f(x)\,dx$$

iii) $f(x) \leq g(x)$



$$A = \int_a^b (g(x) - f(x))\,dx$$

NB: An area is non-negative

Ex: What is the area between $y = x$ and $y = x^2$ in $[0,1]$?

NB: MAKE A FIGURE!

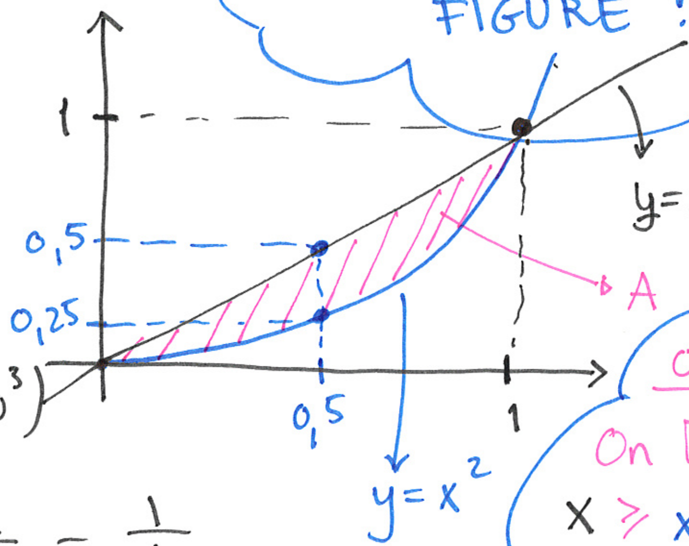
$$A = \int_0^1 (x - x^2)\,dx$$

Case iii)

$$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{x=0}^{1}$$

$$= \frac{1}{2}1^2 - \frac{1}{3}1^3 - \left(\frac{1}{2}0^2 - \frac{1}{3}0^3\right)$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \underline{\underline{\frac{1}{6}}} \quad (\approx 0{,}167)$$

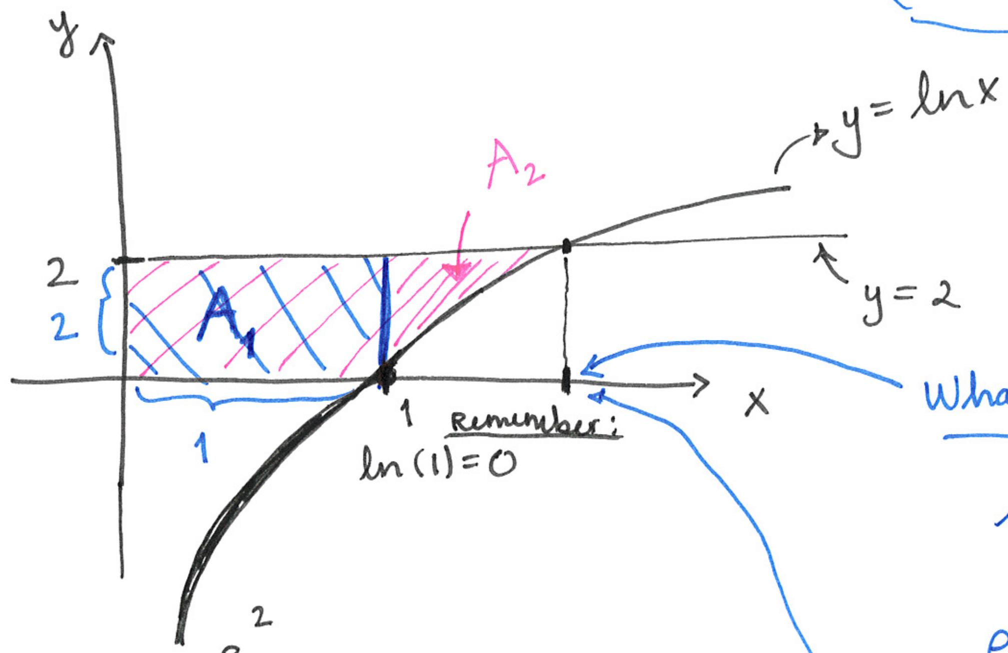


OBS: On $[0,1]$ $x \geq x^2$

START 11.00

Ex: Area bounded by $y = \ln x$, $y = 2$, the y-axis and the x-axis?

MAKE A FIGURE!



$A_1 = 1 \cdot 2 = \underline{2}$

What is this point?

$\ln x = 2$

$e^{\ln x} = e^2$

$\boxed{x = e^2}$

$$A_2 = \int_1^{e^2} (2 - \ln x)\, dx$$

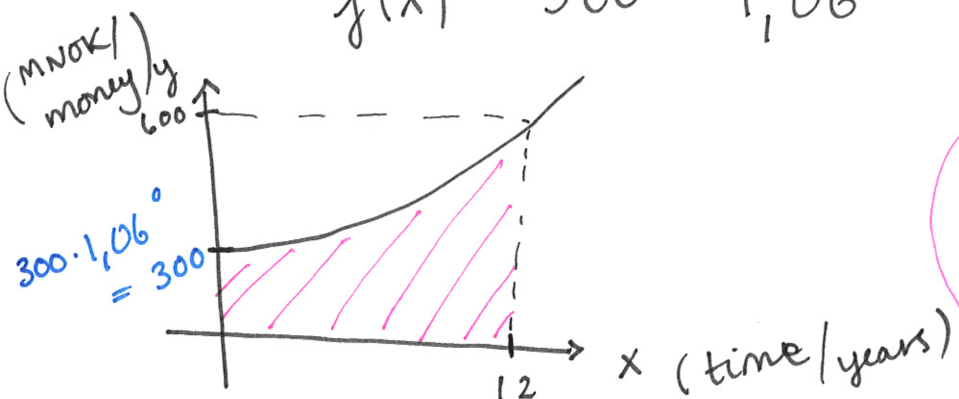
$$\text{Area} = A_1 + A_2 = 2 + \int_1^{e^2} (2 - \ln x)\, dx$$

$= \overset{\text{(see scan)}}{\ldots} = \underline{\underline{e^2 - 1}} \ (\approx 6{,}389)$

Economic applications of the definite integral

Ex: Continuous cash flow:

$f(x) = 300 \cdot 1{,}06^x$ (Cash flow in MNOK / year)



Rule of 72: Doubling takes approx $\frac{72}{6} = 12$ years when 6% interest per year

Total cash flow in 12 years = the area under the graph in [0, 12]

$$= \int_0^{12} f(x)\,dx = \int_0^{12} 300 \cdot 1{,}06^x\,dx \longrightarrow \int a^x dx$$

FORMULA from 1st integration lecture

$$= \left[300 \, \frac{1{,}06^x}{\ln(1{,}06)}\right]_{x=0}^{12}$$

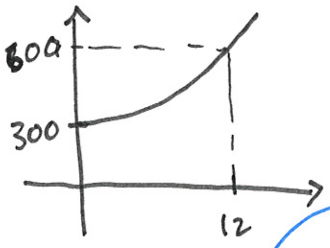
$$= \frac{300}{\ln(1{,}06)} \left[1{,}06^x\right]_{x=0}^{12}$$

$$= \frac{300}{\ln(1{,}06)} \left(1{,}06^{12} - \underbrace{1{,}06^0}_{1}\right)$$

$$= \frac{300}{\ln(1{,}06)} \left(1{,}06^{12} - 1\right) \approx \underline{5\,211 \text{ MNOK}}$$

Net present value of a continuous cash flow

NPV



$f(x) = 300 \cdot 1{,}06^x$, cash flow

ASSUME: r = discount rate = 10 %, continuous discounting

Money in the future is worth less than money today: Inflation

NPV: $\displaystyle\int_0^{12} \underbrace{f(x)}_{\text{cash flow}} \underbrace{e^{-rx}}_{\text{continuous discounting}} dx = \int_0^{12} 300 \cdot 1,06^{x} \cdot e^{-0,10x} dx$

$$= 300 \int_0^{12} 1,06^{x} e^{-0,10x} dx$$

$$= 300 \int_0^{12} e^{\ln(1,06)x} e^{-0,10x} dx$$

$$e^{\ln(1,06)x} = \left(e^{\ln(1,06)}\right)^{x} = (1,06)^{x} = 1,06^{x}$$

$$= 300 \int_0^{12} e^{(\ln(1,06) - 0,10)x} dx$$

Substitution:

$$u = (\ln(1,06) - 0,10)x$$

$$du = (\ln(1,06) - 0,10)dx$$

$$\int e^{(\ln(1,06) - 0,10)x} dx$$

$$= \int e^{u} \frac{1}{\ln(1,06) - 0,10} du$$

$$= \frac{1}{\ln(1,06) - 0,10} e^{u} + C$$

$$= 300 \left[\frac{1}{\ln(1,06) - 0,1} e^{(\ln(1,06) - 0,1)x}\right]_{x=0}^{12}$$

$$= \frac{300}{\ln(1,06) - 0,1}\left(e^{(\ln(1,06) - 0,1)12} - \underbrace{e^{0}}_{1}\right)$$

THIS SAYS 12

$$\approx \underline{\underline{2832 \text{ MNOK}}}$$

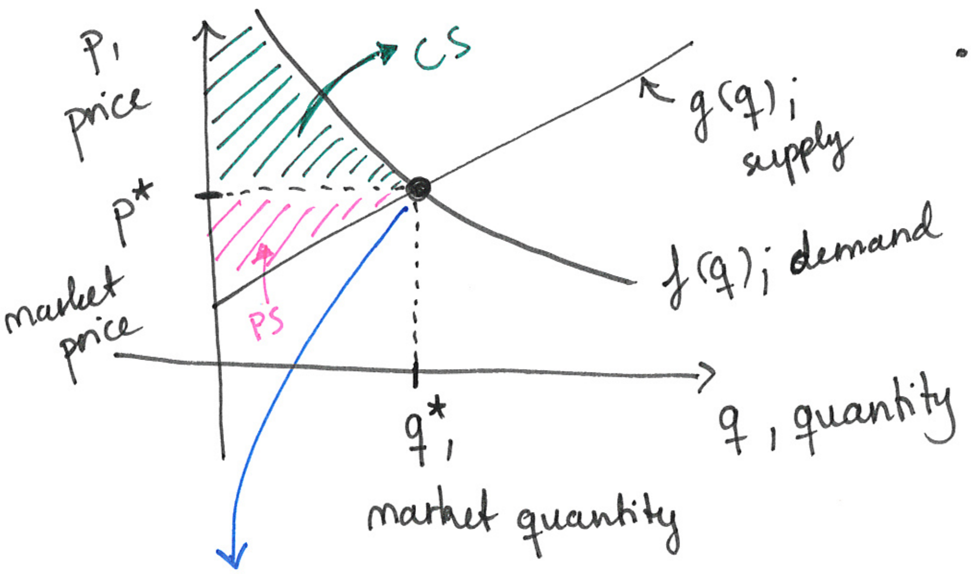
FORMULAS (Economic applications of definite integrals)

Total cash flow: $\displaystyle\int_0^T \underbrace{f(x)}_{f(x):\ \text{cash flow per time unit}} dx$

NPV of cash flow: $\displaystyle\int_0^T f(x)\, e^{-\overbrace{r}^{r:\ \text{discount rate}}\, x} dx$

Consumer / producer surplus

(CS) (PS)



- $g(q)$; supply function (inverse)
- $f(q)$; demand function (inverse)

$$g(q^*) = f(q^*)$$

CS: Consumer surplus: $\displaystyle CS = \int_0^{q^*} f(q) - p^*\, dq$

PS: Producer surplus: $\displaystyle PS = \int_0^{q^*} p^* - g(q)\, dq$