Warm-up: $\displaystyle\int \frac{5}{4-9x^2}\,dx$

What's the plan to solve this integral?

Partial fractions: $4 - 9x^2 = (2-3x)(2+3x)$

$$4 = 2^2, \quad 9x^2 = (3x)^2$$

$$\text{integral} = 5\int \frac{1}{(2-3x)(2+3x)}\,dx$$

$$\frac{1}{4-9x^2} = \frac{1}{(2-3x)(2+3x)} = \frac{A}{2-3x} + \frac{B}{2+3x}$$

$\vdots$

Get 2 linear eqns. with 2 unknowns:

$A = \ldots, \quad B = \ldots$

Then substitution and ln-antidiff.

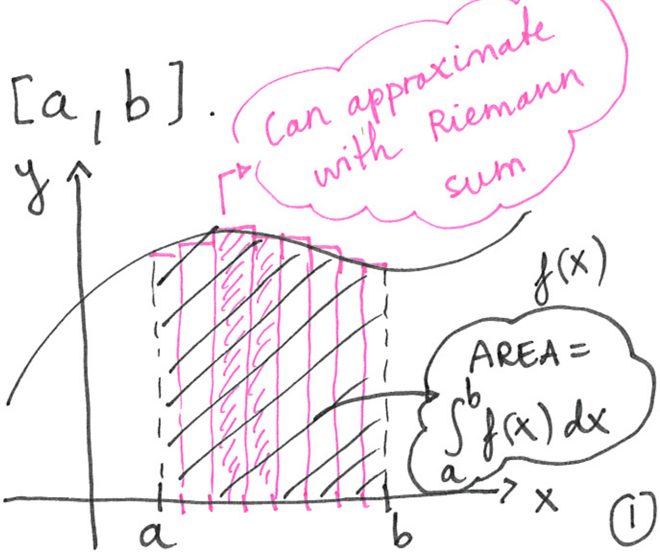
Definite integrals

Assume: i) f is a continuous function on $[a,b]$.

ii) $f(x) \geq 0$ for all x in $[a,b]$.

iii) $a \leq b$.

Then, $\displaystyle\int_a^b f(x)\,dx =$ the area under the graph of f in $[a,b]$



"Def" (definite integral):

TO CALCULATE

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where}$$

$$F'(x) = f(x).$$

So F is antiderivative of f

Why-ish?

$$\int_a^b g'(x)\,dx = g(b) - g(a)$$

$\int_a^b F'(x) = F(b) - F(a)$

Approx: Riemann

$$\sum_a^b \underbrace{\underbrace{g'(x)}_{\text{height of rectangle}}\ \underbrace{\Delta X}_{\text{width of rectangle}}}_{\text{area of rectangle}} = g(b) - g(a)$$

Sum: Add up areas of rectangles

NOTE: Def. of derivative:

$$g'(x) \approx \frac{g(x+\Delta X) - g(x)}{\Delta X}$$

Delta

$$g'(x)\,\Delta X \approx g(x+\Delta X) - g(x)$$

$$\sum_a^b g(x+\Delta X) - g(x) = g(b) - g(a)$$

" sum of lots of small changes = total change "

Could also write out sum and get cancelling $\blacksquare$

Ex:

$$\int_0^1 x^2\,dx = \Big[\underbrace{\tfrac{1}{3}x^3}_{F(x)} + C\Big]_{x=0}^{1}$$

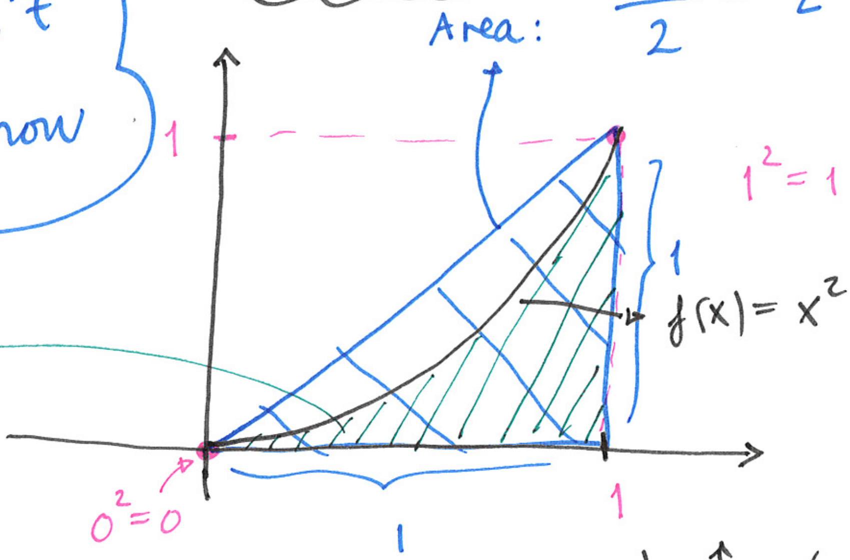
$\int_0^1 f(x)\,dx = F(1) - F(0)$ where F is antiderivative of f

$$= \underbrace{\left(\frac{1}{3}\cdot 1^3 + C\right)}_{F(1)} - \underbrace{\left(\frac{1}{3}\cdot 0^3 + C\right)}_{F(0)}$$

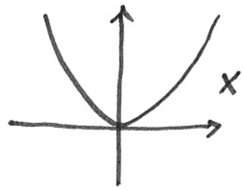
$$= \frac{1}{3} + \underbrace{\cancel{C} - \cancel{C}}\ = \underline{\underline{\frac{1}{3}}}$$

Cancellation of constant always happens: won't write down from now

FIGURE



Area: $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3} < \frac{1}{2}$ = Area of blue triangle



Ex: $$\int_{x=0}^{x=1} x\sqrt{x^2+1}\,dx = \int_{u=1}^{u=2} \cancel{x}\sqrt{u}\,\frac{1}{2\cancel{x}}\,du$$

$\boxed{\text{START } 11.00}$

$u = x^2+1$

$du = 2x\,dx$

$dx = \frac{1}{2x}\,du$

$x = 0 \Rightarrow u = x^2+1 = 0^2+1 = 1$

$x = 1 \Rightarrow u = x^2+1 = 1^2+1 = 2$

$$= \int_1^2 \frac{1}{2}\underbrace{\sqrt{u}}_{u^{\frac{1}{2}}}\,du$$

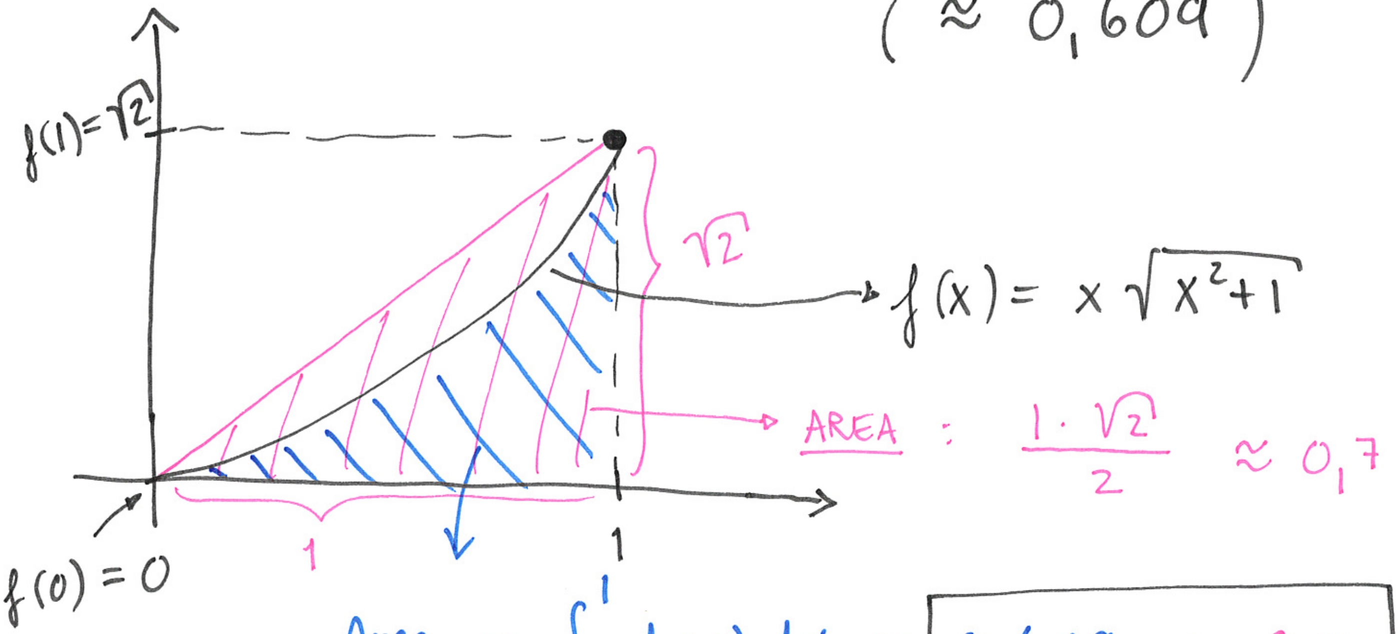
$$= \left[\frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{u=1}^{2}$$

$$= \left[\frac{1}{\cancel{2}} \frac{\cancel{2}}{3} u^{\frac{3}{2}} \right]_{u=1}^{2} = \frac{2\sqrt{2}}{3} - \frac{1\sqrt{1}}{3}$$

$$u^{1+\frac{1}{2}} = u^1 u^{\frac{1}{2}} = u\sqrt{u}$$

$$= \frac{1}{3}\left(2\sqrt{2} - 1\right)$$

$$(\approx 0,609)$$



$$\text{Area} = \int_0^1 f(x)\,dx \approx \boxed{0,609 < 0,7}$$

(Makes sense!)

NOTE: Mind the integration bounds when doing substitution!

Alternative: Instead of inserting into:

$$\left[\frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] = \left[\frac{1}{2} \frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{x=0}^{x=1}$$

sub. back in for x

→ OBS!

Theorem: If f is a continuous function on $[a,b]$ such that $f(x) \geq 0$ for x in $[a,b]$, then:

$$\text{the area under the graph of } f(x) \text{ in } [a,b] \overset{\text{Will "prove"}}{=} \int_a^b f(x)\,dx \underset{\text{Have "shown"}}{=} F(b) - F(a)$$

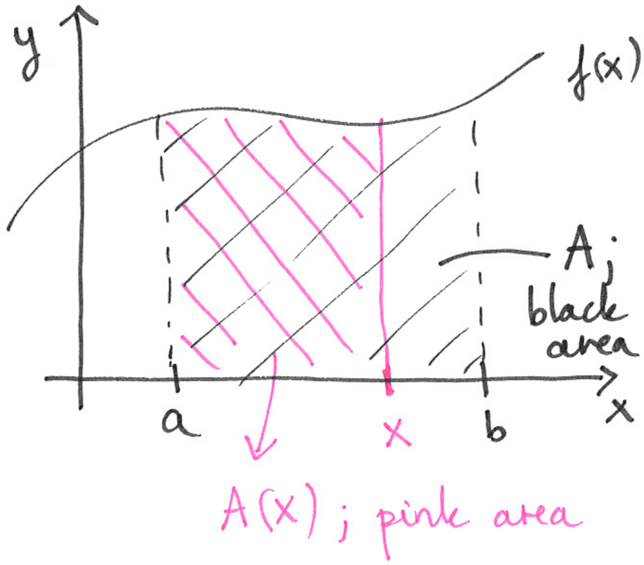
where $F'(x) = f(x)$, so F is an anti-derivative of f.

Why? "Proof" of the first equality:

Define $\underbrace{A(x)}_{\text{area function}} :=$ the area under $f(x)$ for $[a, x]$

Also, let

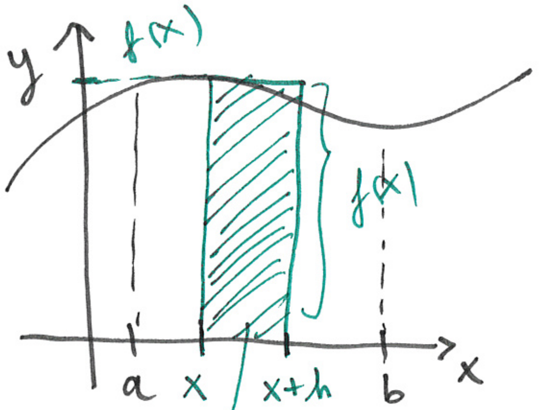
$A =$ area under $f(x)$ in $[a,b]$



FACTS:
- $A(a) = 0$
- $A(b) = A$

→ From the def. of the derivative: $A'(x) \approx \dfrac{A(x+h) - A(x)}{h}$

$$\approx \frac{\text{area of strip/rectangle}}{h}$$



Area of rectangle: $f(x) \cdot h$

OR $\approx A(x+h) - A(x)$

$$= \frac{f(x)\cdot \cancel{h}}{\cancel{h}} = f(x)$$

So: $A'(x) \approx f(x)$, hence $A(x)$ is an antiderivative of $f(x)$.

But then,

$$\int_a^b f(x)\,dx = \Big[A(x) \Big]_{x=a}^{b} = A(b) - A(a)$$

$$= A - 0 = \underbrace{A}_{\text{area under the graph between a and b}}$$

From FACTS

■

Improper integrals

What if: 1) $f(x)$ is not continuous on $[a, b]$?

OR

2) $a = -\infty$ or $b = \infty$?

Ex: $\displaystyle\int_0^1 \underbrace{\frac{1}{x^2}}_{} \,dx \overset{\text{THIS MEANS}}{=} \lim_{a \to 0^+} \int_a^1 \frac{1}{x^2}\,dx$

$\frac{1}{x^2}$ is not defined for $x = 0$