EBA 1180, lecture 21, 11 Nov. 2025, Runar Ile

Plan. 1. Repetition (problems from last week)

1d) implicit differentiation

2) implicitly defined curves

6b) concave/convex functions

8b) convex optimization

2. l'Hôpital's rule

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1. Repetition

1d) Implicit differentiation

$$x^3 - 3xy + y^2 = 0 \qquad (*)$$

We find an expression for y' in y and x by differentiating each side of $(*)$ w.r.t. x and the solve for y'. Here we think of y as a function of x.

Partial calculations

$$(x \cdot y)'_x \overset{\text{prod. rule}}{=} (x)'_x \cdot y + x \cdot (y)'_x$$

$$= 1 \cdot y + x \cdot y' = y + xy'$$

$$(y^2)'_x \overset{\text{chain rule}}{=} 2y \cdot y'_x .$$ Then $(*)$ gives:

$$3x^2 - 3(y + xy') + 2yy' = 0$$

solve this eq. for y':

$$3x^2 - 3y - 3xy' + 2yy' = 0$$

$$(2y - 3x)y' = 3(y - x^2) \quad \Big| : (2y-3x)$$

$$y' = \frac{3(y - x^2)}{2y - 3x} \qquad (**)$$

- Assume $\underline{x=2}$, then we find the possible y-values by solving $(*)$ with $x=2$:

$$2^3 - 3\cdot 2\cdot y + y^2 = 0$$
$$y^2 - 6y = -8$$
$$(y-3)^2 = -8+9 = 1$$

so either $y-3=1$ or $y-3=-1$

that is $\underline{\underline{y=4}}$, $\underline{\underline{y=2}}$

- We use the point-slope formula to find two tangent functions through the points $(2,4)$, resp. $(2,2)$.

$(2,4)$: $\quad y' = \dfrac{3(4-2^2)}{2.4-3.2} = 0$ so the tangent function is constant: $\underline{\underline{h_1(x) = 4}}$

$(2,2)$: $\quad y' = \dfrac{3\cdot(2-2^2)}{2\cdot 2-3\cdot 2} = \dfrac{3\cdot(-2)}{-2} = 3$

so the point-slope formula gives

$$h_2(x) - 2 = 3(x-2)$$
$$\underline{\underline{h_2(x) = 3x-4}}$$

<u>Probl. 2</u> Elimination is the strategy.

(1) In 1a, c and d we got <u>two</u> y-values for <u>one</u> x-value. So 1a, c and d cannot be the blue one (to the right)

so <u>1b has to be the blue one</u>

(2) The red (to the left) and the green (at the bottom) are symmetric around horizontal lines, so their tangents are symmetric too. In particular the slopes are only changing signs (for fixed x-values) This is the case for 1a and 1c.

so <u>1d has to be the purple one</u> (in the middle)

(3) In 1a we have both y-values positive
In 1c one y-value is negative.
If the thicker horizontal lines are the x-axes, then

<u>1a has to be the green graph</u> (at the bottom)

and <u>1c ——— " ——— red graph</u> (upper left)

6b) $f(x) = \ln(x^2 - 2x + 2) - \frac{x}{4} + 1$

<u>Note</u> $x^2 - 2x + 2 = (x-1)^2 + 1 \geq 1$ so $f(x)$ is defined for all values of x.

$$f'(x) = [\ln(x^2 - 2x + 2)]' - \frac{1}{4} + 0$$

Chain rule with

$u = x^2 - 2x + 2$ and $g(u) = \ln(u)$

$u'(x) = 2x - 2$ $\qquad g'(u) = \frac{1}{u}$

$$= \frac{2x-2}{x^2-2x+2} - \frac{1}{4}$$

$$f''(x) = \frac{(2x-2)'(x^2-2x+2) - (2x-2)(x^2-2x+2)'}{(x^2-2x+2)^2}$$

$$= \frac{2\cdot(x^2-2x+2) - \overbrace{(2x-2)(2x-2)}^{4x^2-8x+4}}{(x^2-2x+2)^2}$$

$$= \frac{2x^2 - 4x + 4 - 4x^2 + 8x - 4}{(x^2-2x+2)^2}$$

$$= \frac{-2x^2 + 4x}{(x^2-2x+2)^2} = \frac{-2x(x-2)}{[(x-1)^2+1]^2}$$

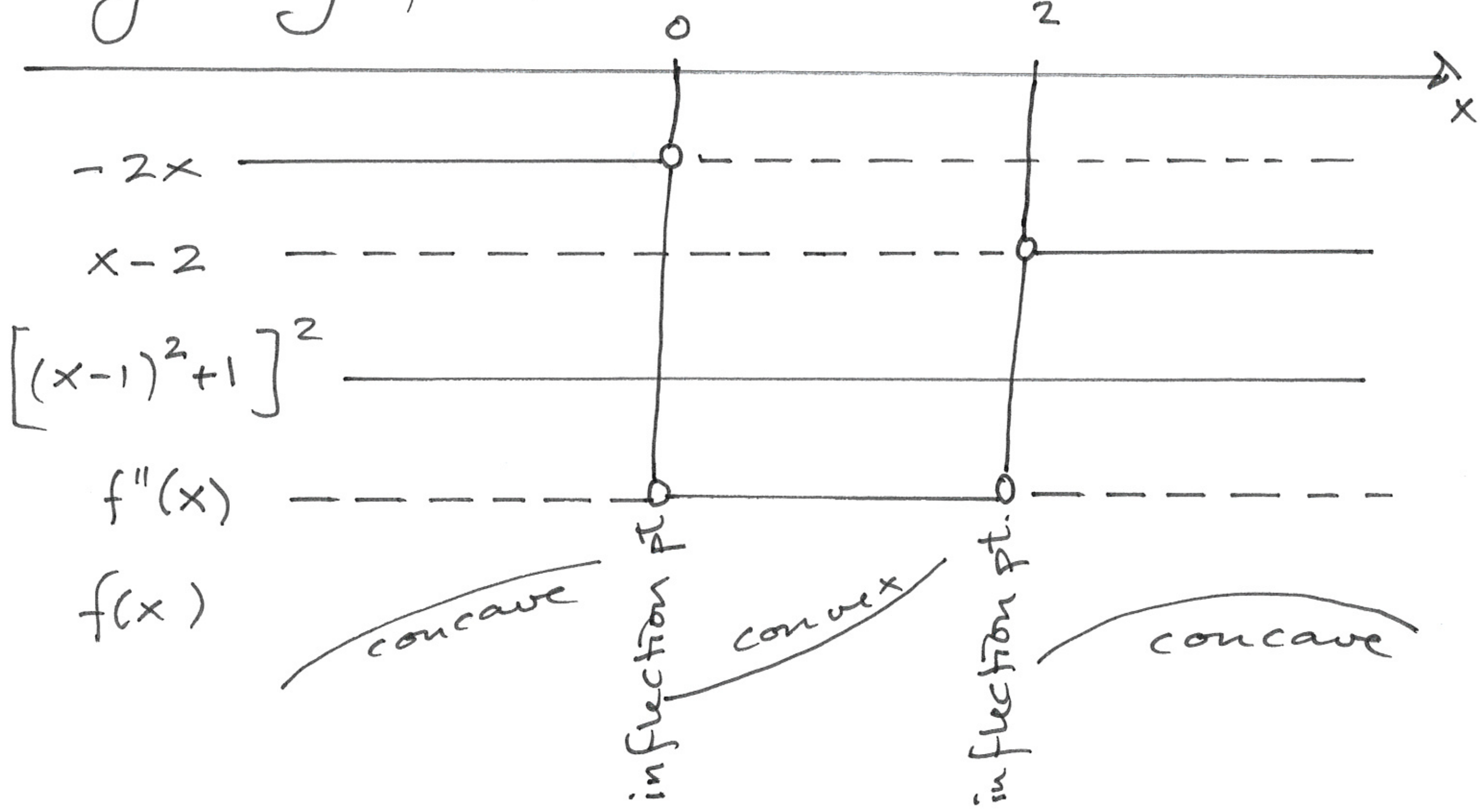
Solve the eq. $f''(x) = 0$

that is $-2x(x-2) = 0$

so $-2x = 0$ or $x - 2 = 0$

$\underline{\underline{x = 0}}$ or $\underline{\underline{x = 2}}$

Sign diag. for $f''(x)$.



<u>Conclusion</u>

$f(x)$ is concave in $\langle \leftarrow, 0]$

— " — convex in $[0, 2]$

— " — concave in $[2, \rightarrow\rangle$

The inflection points for $f(x)$

are $\underline{\underline{x=0}}$ and $\underline{\underline{x=2}}$

because $f''(x)$ changes sign here.

Start: 11.08

Prob. 8b
- Determine the (loc.) max/min. point for $f(x)$.
- Explain why they give global max./min.
- calculate max/min. of $f(x)$.

$$f(x) = \frac{-1}{x(x-6)}, \quad D_f = \langle 0, 6 \rangle$$

- $f'(x) \overset{\text{frac. rule}}{=} \dfrac{2x-6}{[x(x-6)]^2}$. Stationary points: solutions of the eq.

$$f'(x) = 0$$

that is $2x - 6 = 0$ (since $x(x-6) \neq 0$ in D_f)

$$\underline{x = 3}$$

Numerator of $f'(x)$ is changing sign from $-$ to $+$ at $x = 3$ and $[x(x-6)]^2 > 0$

so $f'(x)$ changes sign from $-$ to $+$ at $x=3$.

and $\underline{\underline{x = 3}}$ is a (loc.) min. point.

- Calculate $f''(x) = \left[\dfrac{2x-6}{x^2(x-6)^2}\right]'$

$$\left[x^2 \cdot (x-6)^2\right]' = \underline{2x} \cdot (\underbrace{x-6}) ^2 + \underline{x^2} \cdot \underline{2} \cdot (\underbrace{x-6}) \cdot 1$$

$$= 2x(x-6)(x-6+x)$$

$$= 2x(x-6)(2x-6)$$

$$f''(x) = \frac{2\cdot x^{\cancel{2}}(x-6)^{\overset{1}{\cancel{2}}} - (2x-6)\cdot 2\cancel{x}\cancel{(x-6)}(2x-6)}{x^{\overset{3}{\cancel{4}}}(x-6)^{\overset{3}{\cancel{4}}}}$$

$$= \frac{2x(x-6) - (2x-6)^2\cdot 2}{x^3(x-6)^3}$$

$$= \frac{2(-3x^2+18x-36)}{x^3(x-6)^3} = \frac{-6(x^2-6x+12)}{x^3(x-6)^3}$$

$$= \frac{-6\left[(x-3)^2+3\right]}{x^3(x-6)^3}$$

For $x \in \langle 0, 6\rangle$, $x^3 > 0$ and $(x-6)^3 < 0$

so $f''(x) = \dfrac{\text{neg.}\cdot\text{pos}}{\text{pos}\cdot\text{neg}} = \dfrac{\text{neg}}{\text{neg}} > 0$ for $x \in D_f = \langle 0, 6)$

Hence $f(x)$ is convex in D_f and

$x = 3$ is a global minimum point.

- Minimal value of $f(x)$ is

$$f(3) = \frac{-1}{3(3-6)} = \frac{-1}{-9} = \underline{\underline{\frac{1}{9}}}$$

No maximal value:

$$f(x) \xrightarrow[x\to 0^+]{} +\infty \text{ and } f(x) \xrightarrow[x\to 6^-]{} +\infty$$

2. l'Hôpital's rule

It is about limits of the type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Notation $\lim_{x\to 5} f(x)$ is the number which $f(x)$ is approaching when x is approaching 5.

Ex $f(x) = \dfrac{3x-3}{\ln(x)}$. Want to find $\lim_{x\to 1} f(x)$.

$$\left.\begin{array}{ll}\text{Numerator: } 3x-3 \xrightarrow[x\to 1]{} 3\cdot 1 - 3 = 0 \\ \text{Denominator: } \ln(x) \xrightarrow[x\to 1]{} \ln(1) = 0\end{array}\right\} \frac{0}{0}\text{-expr.}$$

Then we can use l'Hôpital's rule:

$$\lim_{x\to 1} f(x) \overset{\text{l'Hôp}}{=} \lim_{x\to 1} \frac{(3x-3)'}{[\ln(x)]'} = \lim_{x\to 1} \frac{3}{\left(\frac{1}{x}\right)} = \frac{3}{\left(\frac{1}{1}\right)} = \underline{\underline{3}}$$

Check: $f(1.01) = \dfrac{3\cdot 1.01 - 3}{\ln(1.01)} = 3.050$

$f(0.99) = \dfrac{3\cdot 0.99 - 3}{\ln(0.99)} = 2.9850$

Note: Has to be $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$!

Then differentiate numerator and denominator seperately and try to find the limit of the new fraction.

Ex $\lim_{x \to 0} \frac{3x}{e^x - 1}$ $\left(\frac{3 \cdot 0}{e^0 - 1} = \frac{0}{0} \right)$

$\overset{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{3}{e^x} = \frac{3}{e^0} = \frac{3}{1} = \underline{\underline{3}}$

Ex $\lim_{x \to \infty} \frac{x^2}{e^x} \overset{\text{l'Hôp}}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{\text{l'Hôp}}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$

" $\frac{\infty}{\infty}$ " " $\frac{\infty}{\infty}$ "

Meaning : $f(x) = \frac{x^2}{e^x}$ has $y = 0$ as horizontal asymptote.