Problem Sheet 9
DRE 7007 Mathematics

BI Norwegian Business School

Solutions Problem Sheet 9

1. Use dynamic programming:

$$J_3(x) = \max_{u \in U} (3-u)x^2 \qquad \underline{u_3 = 1}$$

$$= \underline{2x^2}$$

$$J_2(x) = \max_{u \in U} (3-u)x^2 + 2(ux)^2 \qquad \underline{u_2 = 3}$$

$$= \max_{u \in U} 3x^2 - ux^2 + 2u^2x^2$$

$$= \underline{18x^2}$$

$$J_1(x) = \max_{u \in U} (3-u)x^2 + 18(ux)^2 \qquad \underline{u_1 = 3}$$

$$= \max_{u \in U} 3x^2 - ux^2 + 18u^2x^2$$

$$= \underline{162x^2}$$

$$J_0(x) = \max_{u \in U} (3-u)x^2 + 162(xu)^2 \qquad \underline{u_0 = 3}$$

$$= \max_{u \in U} 3x^2 - ux^2 + 162u^2x^2$$

$$= \underline{1458x^2}$$

<u>Solution:</u> $V = J_0(x_0) = 1458x_0^2 = \underline{1458}$ with $\begin{cases} u_0 = 3 \\ u_1 = 3 \\ u_2 = 3 \\ u_3 = 1 \end{cases}$

2. Bellman equation:

$$J(x) = \max_{u \in U} \left(-cx^2 - u^2 + \beta \cdot J(x+u) \right)$$

Solutions of the form $J(x) = -Ax^2$:

$$\max_{u \in U} \; -cx^2 - u^2 + \beta \cdot (-A) \cdot (x+u)^2$$

$$= -cx^2 - \frac{(\beta A)^2}{(1+\beta A)^2} x^2 - \beta A \left(1 - \frac{\beta A}{1+\beta A}\right)^2 x^2$$

$$= x^2 \cdot \left(-c - \frac{\beta^2 A^2}{(1+\beta A)^2} - \frac{\beta A}{(1+\beta A)^2} \right)$$

$$= x^2 \cdot \left(-c - \frac{\beta A \cdot \cancel{(\beta A + 1)}}{(1+\beta A)^{\cancel{2}}} \right)$$

$$= x^2 \cdot \left(-c - \frac{\beta A}{1+\beta A} \right)$$

derivative:

$$-2u - \beta A \cdot 2(x+u) = 0$$

$$u(-2 - 2\beta A) = 2\beta A x$$

$$u = \frac{2\beta A x}{-2 - 2\beta A}$$

$$u^* = -\frac{\beta A}{1+\beta A} \cdot x$$

double derivative:

$$-2 - 2\beta A < 0$$

$\Downarrow$

max at u^*

Bellman eqn:

$$-Ax^2 = x^2 \left(-c - \frac{\beta A}{1+\beta A} \right)$$

$$A = c + \frac{\beta A}{1+\beta A}$$

$$A(1+\beta A) = c(1+\beta A) + \beta A$$

$$\beta A^2 + \underbrace{(1 - c\beta - \beta)}_{1-\beta(c+1)} A - c = 0$$

$$A = \frac{-1+\beta(c+1)}{2\beta} \pm \frac{\sqrt{(1-\beta(c+1))^2 - 4\beta \cdot (-c)}}{2\beta}$$

$$= -\frac{1}{2\beta} + \frac{c+1}{2} \pm \sqrt{\left(\frac{1}{2\beta} - \frac{c+1}{2}\right)^2 + \frac{4c}{\beta}}$$

$$\underline{\underline{A = -\frac{1}{2\beta} + \frac{c+1}{2} + \sqrt{\left(\frac{1}{2\beta} - \frac{c+1}{2}\right)^2 + \frac{4c}{\beta}}}}$$

unique positive solution for A