Problem Sheet 6
DRE 7007 Mathematics

BI Norwegian Business School

Solution Problem Sheet 6

1. $\min f = 2x^2+y^2+3z^2$ subj. to $\begin{cases} x-y+2z \geq 3 \\ x+y \geq 3 \end{cases}$

$$\Updownarrow$$

$\max -f = -2x^2-y^2-3z^2$ — " — $\begin{cases} -x+y-2z \leq -3 \\ -x-y \leq -3 \end{cases}$

$$L = -2x^2-y^2-3z^2-\lambda_1(-x+y-2z)-\lambda_2(-x-y)$$

FOC: $L'_x = -4x+\lambda_1+\lambda_2 = 0$ $\Rightarrow \lambda_2 = 4x-\lambda_1 = 4x-3z$

$L'_y = -2y-\lambda_1+\lambda_2 = 0$ $\quad \lambda_2 = 2y+\lambda_1 = 2y+3z$ $\Bigg\}$ $4x-3z = 2y+3z \Rightarrow 6z = 4x-2y$, $3z = 2x-y$

$L'_z = -6z+2\lambda_1 = 0 \Rightarrow \lambda_1 = 3z$

$\Downarrow$

$\lambda_1 = 2x-y$

$\lambda_2 = 2x+y$

$z = \frac{2}{3}x-\frac{1}{3}y$

$\begin{rcases} x-y+2z > 3 \\ x+y > 3 \end{rcases}$ $\lambda_1=\lambda_2=0 \Rightarrow 2x-y=2x+y=0$, $x=y=0$, $z=0$ $\Rightarrow x+y>3$, $0>3$ contradiction

$\begin{rcases} x-y+2z>3 \\ x+y=3 \end{rcases}$ $\lambda_1=0$, $\lambda_2 \geq 0$ $\Rightarrow 2x-y=0$, $x+y=3$ $\Rightarrow y=2x$, $3x=3$, $x=1$ $\Rightarrow x=1$, $y=2$, $z=0$; $\lambda_1=0$, $\lambda_2=4$ $\Rightarrow x-y+2z>3$, $-1>3$ contradiction

$x-y+2z=3$, $x+y>3$; $\lambda_1 \geq 0$, $\lambda_2=0$ $\Rightarrow 2x+y=0$, $y=-2x$, $z=4/3\,x$ $\Rightarrow x-y+2z=3$, $x+2x+8/3\,x=3$, $17/3\,x=3$, $x=9/17$ $\Rightarrow x=9/17$, $y=-18/17$, $z=\ldots$ $\Rightarrow x+y>3$, $-9/17>3$ contradiction

$x-y+2z=3$, $x+y=3$; $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ $\Rightarrow y=3-x$, $z=\frac{2}{3}x-\frac{1}{3}(3-x) = x-1$ $\Rightarrow x-y+2z=3$, $x-(3-x)+2(x-1)=3$, $4x=3+5=8$, $x=2$ $\Rightarrow x=2$, $y=1$, $z=1$, $\lambda_1=3$, $\lambda_2=5$ — Solution

$\mathcal{L}(x,y,z;3,5) = -2x^2-y^2-3z^2-3(-x+y-2z)-5(-x-y)$ (strict) concave $\Rightarrow$ $\underline{\underline{x=2, y=1, z=1}}$ maximizer of $-f$ = minimizer of f

For problem with $f = (xy+x+y+1)e^{x+y+2}$ see next page.

2. $f = (xy-x-y+1)e^{x+y-2}$ subj. to $x^2+y^2=1$

$= (x-1)(y-1)\, e^{x+y-2}$

$L = (x-1)(y-1)e^{x+y-2} - \lambda(x^2+y^2)$

FOC: $L'_x = x(y-1)\,e^{x+y-2} - \lambda\cdot 2x = 0 \qquad x=0 \text{ or } 2\lambda = (y-1)e^{x+y-2}$

$L'_y = (x-1)y\, e^{x+y-2} - \lambda\cdot 2y = 0 \qquad y=0 \text{ or } 2\lambda = (x-1)e^{x+y-2}$

C: $x^2+y^2=1$

$x=0, y=0$: $x^2+y^2 = 0 \neq 1$.

$x=0, y\neq 0$: $x^2+y^2=1 \Rightarrow y=\pm 1,\ \lambda = -\frac{1}{2}e^{\pm 1-2} \Rightarrow (0,1; e^{-1})$ $f=0$

$(0,-1, e^{-3})$ $f = 2e^{-3}$

$x\neq 0, y=0$: $x^2+y^2=1 \Rightarrow x=\pm 1,\ \lambda = -\frac{1}{2}e^{\pm 1-2} \Rightarrow (1,0; e^{-1})$ $f=0$

$(-1,0; e^{-3})$ $f = 2e^{-3}$

$x\neq 0, y\neq 0$: $(y-1)\,e^{x+y-2} = (x-1)\,e^{x+y-2}$

$y-1 = x-1$

$y = x \Rightarrow x=y=\pm\frac{\sqrt{2}}{2} \Rightarrow \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}; \left(\frac{1}{2}\sqrt{2}-1\right)e^{\sqrt{2}-2}\right),\ f = \left(\frac{1}{2}\sqrt{2}-1\right)^2 e^{\sqrt{2}-2}$

$\left(-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}; \left(-\frac{1}{2}\sqrt{2}-1\right)e^{-\sqrt{2}-2}\right),\ f = \left(-\frac{1}{2}\sqrt{2}-1\right)^2 e^{-\sqrt{2}-2}$

$\left.\begin{array}{l}\text{D closed, bounded}\\ \text{f cont.}\end{array}\right\} \Rightarrow$ There is min and max $\Rightarrow$ Min: $\underline{f=0 \text{ at } (0,1) \text{ and } (1,0)}$

NDCQ satisfied. Max: $\underline{f = 2e^{-3} \text{ at } (0,-1) \text{ and } (-1,0)}$

<u>If the constraint is changed to $x^2+y^2 \leq 1$:</u>

D still compact, so there is a min and max. $\Rightarrow$ Either on $x^2+y^2=1$ or $x^2+y^2<1$

NDCQ satisfied.

<u>$x^2+y^2<1$</u>: Only unconstrained optima are possible $\Rightarrow f'_x = x(y-1)e^{x+y-2} = 0 \Rightarrow x=0 \text{ or } y=1$

$f'_y = (x-1)y\, e^{x+y-2} = 0 \qquad x=1 \text{ or } y=0$

$\Downarrow$

$(0,0)$ $x^2+y^2<1$ ok.

$(1,1)$ $x^2+y^2 = 2 \nless 1$

$\Downarrow$

Solution: $(0,0)$

$f(0,0) = e^{-2}$

Since $e^{-2} > 2e^{-3}$, we get:

min: $(x,y) = (0,1), (1,0)$ $\quad f=0$

max: $(x,y) = (0,0)$ $\quad f = e^{-2}$

2. First version (with misprint): max/min $f = (xy + x + y + 1)e^{x+y+2}$ subj. to. $x^2 + y^2 = 1$

$$L = f - \lambda g = (xy + x + y + 1)e^{x+y+2} - \lambda(x^2 + y^2) = (x+1)(y+1)e^{x+y+2} - \lambda(x^2+y^2)$$

$$L'_x = (x+2)(y+1)e^u - \lambda \cdot 2x = 0 \qquad (u = x+y+2)$$

$$L'_y = (x+1)(y+2)e^u - \lambda \cdot 2y = 0$$

$$x^2 + y^2 = 1$$

$\Downarrow$

$(y-x)e^u = 2\lambda(x-y)$

$(y-x)(e^u + 2\lambda) = 0 \qquad x = y$ or $2\lambda = -e^u$

$x = y$: $x = y = \pm\sqrt{1/2} \Rightarrow (\sqrt{\tfrac12}, \sqrt{\tfrac12}, \lambda_1)$ $\quad f = (1 + \sqrt{\tfrac12})^2 \cdot e^{2+\sqrt2}$

$(-\sqrt{\tfrac12}, -\sqrt{\tfrac12}, \lambda_2)$ $\quad f = (1 - \sqrt{\tfrac12})^2 \cdot e^{2-\sqrt2}$

$x \neq y$: $2\lambda = -e^u$, $(x+2)(y+1) + x = 0$

$xy + 2x + 2y + 2 = 0$

$(x+2)(y+2) = 2 \Rightarrow x = \dfrac{2}{y+2} - 2$

$x^2 + y^2 = \left(\dfrac{2}{y+2} - 2\right)^2 + y^2 = \dfrac{4}{(y+2)^2} - 8/(y+2) + 4 + y^2 = 1$

$4 - 8(y+2) + (3 + y^2)(y+2)^2 = 0$

$y^4 + 4y^3 + 7y^2 + 4y = 0$

$y \cdot (y+1)(y^2 + 3y + 4) = 0$

$\underline{y = 0}, \ \underline{y = -1}$

$x = \pm 1, y = 0 \Rightarrow (\pm 1, 0, \lambda_3) \quad f = 0$

$x = 0, y = -1 \Rightarrow (0, -1, \lambda_4) \quad f = 0$

Conclusion:

D compact, f cont $\Rightarrow$ there is global min/max $\Rightarrow$ Min: $f = 0$ at $(\pm 1, 0), (0, -1)$

Max: $f = (1 + \sqrt{\tfrac12})^2 e^{2+\sqrt2}$ at $(\sqrt{\tfrac12}, \sqrt{\tfrac12})$

Changed to $x^2 + y^2 \leq 1$:

On $x^2 + y^2 < 1$, $f'_x = f'_y = 0$ gives $(x+2)(y+1) = 0$ $\quad x = -2$ or $y = -1$

$(x+1)(y+2) = 0$ $\quad x = -1$ or $y = -2$

$\uparrow$ do not satisfy $x^2 + y^2 < 1$

Conclusion: Same max/min as before

3. max/min $f = xy + xz - yz$ subj. to $x^2 + y^2 + z^2 \le 1$

$L = xy + xz - yz - \lambda(x^2 + y^2 + z^2)$

FOC:
$$\left.\begin{aligned} L'_x &= y + z - \lambda \cdot 2x = 0 \\ L'_y &= x - \cancel{z} - \lambda \cdot 2y = 0 \\ L'_z &= x - y - \lambda \cdot 2z = 0 \end{aligned}\right\}$$

$$\begin{pmatrix} -2\lambda & 1 & 1 \\ 1 & -2\lambda & -1 \\ 1 & -1 & -2\lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} -2\lambda & 1 & 1 \\ 1 & -2\lambda & -1 \\ 1 & -1 & -2\lambda \end{vmatrix} = -2\lambda(4\lambda^2 - 1) - 1(-2\lambda + 1) + 1(-1 + 2\lambda)$$
$$= -8\lambda^3 + 2\lambda + 2\lambda - 1 - 1 + 2\lambda$$
$$= -8\lambda^3 + 6\lambda - 2 = 0$$
$$(\lambda + 1)(-8\lambda^2 + 8\lambda - 2) = 0$$
$$\lambda = -1, \quad \lambda = \frac{-8 \pm \sqrt{64 - 64}}{2 \cdot (-8)} = \frac{1}{2}$$

$\Downarrow$

$\lambda \neq -1, 1/2$: $x = y = z = 0 \Rightarrow x^2 + y^2 + z^2 \le 1$ ok. $f = 0$

$\lambda = -1$:
$$\begin{aligned} 2x + y + z &= 0 \\ x + 2y - z &= 0 \\ x - y + 2z &= 0 \end{aligned} \Rightarrow x = -z, \; y = +z \Rightarrow x = \tfrac{1}{3}\sqrt{3}, y = -\tfrac{1}{3}\sqrt{3}, z = -\tfrac{1}{3}\sqrt{3};\ \lambda = -1$$
or
$$x = -\tfrac{1}{3}\sqrt{3}, y = \tfrac{1}{3}\sqrt{3}, z = \tfrac{1}{3}\sqrt{3};\ \lambda = -1$$

$\lambda = 1/2$:
$$\begin{aligned} -x + y + z &= 0 \\ x - y - z &= 0 \\ x - y - z &= 0 \end{aligned} \Rightarrow x = y + z$$
$\Downarrow$
$$(y+z)^2 + y^2 + z^2 \le 1$$
$$2y^2 + 2z^2 + 2yz \le 1$$
$$\underline{y^2 + yz + z^2 \le 1/2}$$

$$f = (y+z)y + (y+z)z - yz$$
$$= y^2 + yz + z^2 = \underline{1/2}$$

D compact, f cont
$\Downarrow$
there is max/min

NDCQ:

rk $(2x \;\; 2y \;\; 2z) = 1$
when $x^2 + y^2 + z^2 = 1$ ok.

Max: $\lambda \ge 0$,
$\lambda = 0$ if $x^2 + y^2 + z^2 < 1$
$\Downarrow$

$(0, 0, 0;\ 0)$ $f = 0$

$(x, y, z;\ 1/2)$ $f = 1/2$

s.t. $x = y + z$
$y^2 + yz + z^2 = 1/2$

$\Downarrow$

$\underline{f = 1/2}$ is max, at
all (x, y, z) s.t. $\begin{cases} x = y + z \\ y^2 + yz + z^2 = 1/2 \end{cases}$

Min: $\lambda \le 0$
$\lambda = 0$ if $x^2 + y^2 + z^2 < 1$

$(0, 0, 0;\ 0)$ $f = 0$

$(\pm\tfrac{1}{3}\sqrt{3}, \mp\tfrac{1}{3}\sqrt{3}, \mp\tfrac{1}{3}\sqrt{3};\ -1)$ $f = -1$

$\Downarrow$

$\underline{f = -1}$ is min, at
$(\tfrac{1}{3}\sqrt{3}, -\tfrac{1}{3}\sqrt{3}, -\tfrac{1}{3}\sqrt{3})$ and $(-\tfrac{1}{3}\sqrt{3}, \tfrac{1}{3}\sqrt{3}, \tfrac{1}{3}\sqrt{3})$