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Solution DRE7017 06/2017

1. $f = xz - y^2$

a) $f'_x = z$, $f'_y = -2y$, $f'_z = x$

$H(f) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$\Delta_2^{13} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$

indef. matrix

$\Downarrow$

f not convex
not concave

b) $L = xz - y^2 - \lambda(x^2 + y^2 + z^2)$

FOC: $\begin{cases} L'_x = z - \lambda \cdot 2x = 0 \\ L'_y = -2y - \lambda \cdot 2y = 0 \\ L'_z = x - \lambda \cdot 2z = 0 \end{cases}$

C: $x^2 + y^2 + z^2 = 6$

Solutions of FOC+C $\Rightarrow$ candidates for max/min

$-2y - \lambda \cdot 2y = 0$

$2y(-1 - \lambda) = 0$

$y = 0$ or $\lambda = -1$

a) $y = 0$: $\left.\begin{matrix} z = 2\lambda x \\ x = 2\lambda z \end{matrix}\right\}$ $z = (2\lambda) x = (2\lambda)^2 z$

$z(1 - 4\lambda^2) = 0$

$z = 0$ or $1 - 4\lambda^2 = 0$

$\lambda^2 = 1/4$

$\lambda = \pm 1/2$

$z = 0 \Rightarrow x = 0 \Rightarrow$
$(x, y, z) = (0, 0, 0)$ not possible
(c) not fulfilled

$\lambda = 1/2 \Rightarrow z = x$, $x^2 + y^2 + z^2 = 6$

$2x^2 = 6$

$x^2 = 3$ $\quad x = \pm\sqrt{3}$

$f = 3 \longrightarrow (x, y, z; \lambda) = (\pm\sqrt{3}, 0, \pm\sqrt{3}; \tfrac{1}{2})$

$\lambda = -1/2 \Rightarrow z = -x$, $2x^2 = 6$

$x^2 = 3$

$x = \pm\sqrt{3}$

$f = -3 \longrightarrow (x, y, z; \lambda) = (\pm\sqrt{3}, 0, \mp\sqrt{3}; -\tfrac{1}{2})$

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b) $\lambda = -1$: $\left.\begin{aligned} z + 2x = 0 \\ x + 2z = 0 \end{aligned}\right\}$ $x = y = 0$, $x^2 + y^2 + z^2 = 6$

$y^2 = 6$

$y = \pm\sqrt{6}$

$(x, y, z; \lambda) = (0, \pm\sqrt{6}, 0; -1)$ $f = -6$

Conclusion:

The problem has max/min since $x^2+y^2+z^2=6$ is bounded (extreme value theorem)

NDCQ fulfilled: $g'_x = g'_y = g'_z = 0$

$2x = 2y = 2z = 0$

$x = y = z = 0$ $\longrightarrow$ not adm: $x^2+y^2+z^2 = 0 \neq 6$

$\Downarrow$

max/min is among candidates with FOC+C fulfilled.

$f = 3$ is max at $(\pm\sqrt{3}, 0, \pm\sqrt{3})$

$f = -6$ is min " $(0, \pm\sqrt{6}, 0)$

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2.

$$x' = y-2$$
$$y' = z-3$$
$$z' = 8x-2y-5z-5$$

a) Steady state $(\bar{x}, \bar{y}, \bar{z})$:

$$\left.\begin{array}{l} x'=0 \\ y'=0 \\ z'=0 \end{array}\right\} \quad \begin{array}{l} y-2=0 \\ z-3=0 \\ 8x-2y-5z-5=0 \end{array}$$

$\underline{y=2}$
$\underline{z=3}$
$8x = 2y+5z+5 = 24$
$\Rightarrow \underline{x=3}$

$$\underline{\underline{\bar{x}=3, \ \bar{y}=2, \ \bar{z}=3}}$$

b) Choose $\underline{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$, i.e.

$$w_1 = x-3$$
$$w_2 = y-2$$
$$w_3 = z-3$$

Then we get:

$$\underline{w}' = \begin{pmatrix} w_1' \\ w_2' \\ w_3' \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$= A \cdot \underline{x} - A\bar{x} = A(\underline{x} - \bar{x}) = A \cdot \underline{w}$$

We use that by construction,

$$A \cdot \underline{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -2 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

We can therefore write the system as

$$\underline{\underline{\underline{w}' = A\underline{w}}} \quad \text{or} \quad \begin{array}{l} w_1' = w_2 \\ w_2' = w_3 \\ w_3' = 8w_1 - 2w_2 - 5w_3 \end{array}$$

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Eigenvalues of A:

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 8 & -2 & -5-\lambda \end{vmatrix} = 0$$

$$-\lambda \cdot \big((-\lambda)(-5-\lambda) - 0\big) - 1 \cdot (2\lambda - 8) = 0$$

$$\lambda^2(-5-\lambda) - 2\lambda + 8 = -\lambda^3 - 5\lambda^2 - 2\lambda + 8 = 0$$

$\lambda=1$: $-1 - 5 - 2 + 8 = 0$ ok $\Rightarrow$ $\lambda=1$ is eigenvalue.

$$-\lambda^3 - 5\lambda^2 - 2\lambda + 8 = (\lambda-1)\cdot(-\lambda^2 - 6\lambda - 8)$$

$$= -(\lambda-1)(\lambda^2+6\lambda+8) = -(\lambda-1)(\lambda+2)(\lambda+4) = 0$$

All eigenvalues: $\lambda=1,\ \lambda=-2,\ \lambda=-4$

c) Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + C_1 \cdot \underline{v_1} e^{t} + C_2 \underline{v_2} e^{-2t} + C_3 v_3 e^{-4t}$$

where $\underline{v_1}, \underline{v_2}, \underline{v_3}$ are eigenvectors for $\lambda = 1, -2, -4$.

$\lambda=1$: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 8 & -2 & -6 \end{pmatrix} \to \dots \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $x = y = z$, $y = z$, z free $\to$ ($z=1$) $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda=-2$: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 8 & -2 & -3 \end{pmatrix} \to \dots \to \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $x = -\frac{1}{2}y = \frac{1}{4}z$, $y = -\frac{1}{2}z$, z free $\to$ $\underline{v_2} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ ($z=4$)

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$\underline{\lambda=-4}$: $\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 8 & -2 & -1 \end{pmatrix} \to \dots \to \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x = -\tfrac{1}{4}y = \tfrac{1}{16}z$, $y = -\tfrac{1}{4}z$, z free $\Rightarrow \underline{v_3} = \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix}$

$z=16$

Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + c_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix} e^{-4t}$$

d)

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix} \leftarrow t=0$$

$$\begin{pmatrix} x_0-3 \\ y_0-2 \\ z_0-3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Stability for $\underline{c_1=0}$ since $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$ when $c_1=0$

$(e^{-2t}, e^{-4t} \to 0$ as $t \to \infty)$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -4 \\ 1 & 4 & 16 \end{pmatrix}^{-1} \begin{pmatrix} x_0-3 \\ y_0-2 \\ z_0-3 \end{pmatrix}$$

$$= \frac{1}{-30} \begin{pmatrix} -16 & -12 & -2 \\ -20 & 15 & 5 \\ 6 & -3 & -3 \end{pmatrix} \begin{pmatrix} x_0-3 \\ y_0-2 \\ z_0-3 \end{pmatrix}$$

$$c_1=0 \Rightarrow -\frac{1}{30}\left(-16(x_0-3) - 12(y_0-2) - 2(z_0-3)\right) = 0 \quad | \cdot 30$$

$$\underline{\underline{16(x_0-3) + 12(y_0-2) + 2(z_0-3) = 0}}$$

Equation for all initial values (x_0, y_0, z_0) s.t. $c_1=0$

For $(x_0,y_0,z_0) = (4,1,1)$, we see that $16 \cdot 1 + 12 \cdot (-1) + 2 \cdot (-2) = 0$, so this gives $\underline{\text{stability}}$.

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3.

a) Since $x_0 = 2$ and $x_{t+1} = x_t(1+u_t)$ with $u_t \in [0,1]$, it follows that

$$x_{t+1} = x_t \cdot (1+u_t) \geq x_t$$

Since $x_0 = 2$, $x_1 \geq x_0$, $x_2 \geq x_1$, ... it follows by induction that $x_t \geq 2$ for $1 \leq t \leq n$.

b) $n=2$: $\min \sum_{t=0}^{2} 1 + x_t^2 - u_t^2$ when $\begin{cases} x_0 = 2 \\ x_{t+1} = x_t(1+u_t) \\ u_t \in U = [0,1] \end{cases}$

$$J_2(x) = \min_{u \in U} 1 + x^2 - u^2 = 1 + x^2 - 1 = \underline{x^2} \quad \text{when } \underline{u = 1}$$

$$J_1(x) = \min_{u \in U} \underbrace{1 + x^2 - u^2 + (x(1+u))^2}_{1 + x^2 - u^2 + x^2(1+2u+u^2)}$$

increasing in u $\rightarrow$ $= \underbrace{1 + 2x^2}_{\text{pos.}} + \underbrace{2x^2 u}_{\text{pos.}} + \underbrace{(x^2-1)u^2}_{\text{pos.}}$ ← since $x \geq 2$ from a)

$$= \underline{1 + 2x^2} \quad \text{when } u = 0$$

$$J_0(x) = \min_{u \in U} \underbrace{1 + x^2 - u^2 + 1 + 2(x(1+u))^2}_{1 + x^2 - u^2 + 1 + 2x^2(1+2u+u^2)}$$

increasing in u $\rightarrow$ $= \underbrace{2 + 3x^2}_{\text{pos.}} + \underbrace{4x^2 u}_{\text{pos}} + \underbrace{(2x^2-1)u^2}_{\text{pos.}}$

$$= \underline{2 + 3x^2} \quad \text{when } u = 0$$

$u_0 = 0$ $x_0 = 2$
$u_1 = 0$ $x_1 = 2$
$u_2 = 1$ $x_2 = 2$

$x_0 = 2 \Rightarrow$ min value $= J_0(x_0) = 2 + 3 \cdot 2^2 = \underline{\underline{14}}$

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4. $(f = p,\ g = q)$

a) $\|f\| = \sup_{x\in[0,1]} |e^x| = e^1 = \underline{\underline{e}}$ (since e^x is incr.)

$\|g\| = \sup_{x\in[0,1]} |2+x| = 2+1 = \underline{\underline{3}}$ (since $2+x$ is incr.)

$d(f,g) = \|f-g\| = \sup_{x\in[0,1]} |e^x-(2+x)| = |e^1-(2+1)| = |e-3| = \underline{\underline{3-e}}$

Since $(e^x-(2+x))' = e^x - 1 \geq 0$ for $x\in[0,1]$ $\Rightarrow$ $e^x-(2-x)$ inc. in $[0,1]$.

b) $d(f,h) = \sup_{x\in[0,1]} |f(x)-h(x)| = 0$

$\Downarrow$

$f(x)-h(x) = 0$ for all $x\in[0,1]$

$\Downarrow$

$f = h$ in V

This means that $h(x) = f(x) = e^x$ is the only element in the set

$$\{h\in V : d(f,h) = 0\}$$