Exam	Final exam in DRE 7017 Mathematics, Ph.D.
Date	October 14th, 2016 at 0900 - 1200

Question 1.

We consider the system of linear differential equations given by

$$\begin{aligned} \dot{x} &= 2x + y + 5z - 9 \\ \dot{y} &= x + y - 3 \\ \dot{z} &= x + z - 2 \end{aligned}$$

(A) Find the steady state $(\overline{x}, \overline{y}, \overline{z})$.
(B) Rewrite the system in the form $\mathbf{w}' = A\mathbf{w}$ and use this to solve the system.
(C) Find all initial states (x_0, y_0, z_0) such that $(x, y, z) \to (\overline{x}, \overline{y}, \overline{z})$ when $t \to \infty$.

Question 2.

We consider the function $h(x, y, z, w) = xw - yz - a(x^2 + 4y^2) - b(4z^2 + 9w^2)$ with parameters $a, b > 0$ defined on $\mathbb{R}^4$.

(A) For which values of the parameters a, b is h concave, and for which values is it convex?
(B) Let $D = \{(x, y, z, w) : x^2 + 4y^2 = 4 \text{ and } 4z^2 + 9w^2 = 36\}$, and find the maximum value

$$\max_{(x,y,z,w)\in D} f(x, y, z, w) = xw - yz$$

if it exists. Justify your answer.

Question 3.

We consider the optimal control problem

$$\max \sum_{t=0}^{4} \left(1 + x_t^2 - u_t^2\right) \quad \text{when} \quad \begin{cases} x_0 = 1 \\ x_{t+1} = x_t + u_t \\ u_t \in U \end{cases}$$

with control region $U = [0, 1]$.

(A) Solve the optimal control problem.
(B) Will the maximal value increase or decrease if x_0 increases? Justify your answer.

Question 4.

We consider the function $f_n(x) = nx(1 - x)^n$ for $n = 1, 2, 3, \ldots$ in the function space $V = C([0, 1])$ of continuous functions on the unit interval $[0, 1]$. We equip V with the sup norm

$$||f|| = \sup_{x\in[0,1]} |f(x)|$$

and the corresponding metric $d(f, g) = ||f - g||$ for $f, g \in V$.

(A) Compute $||f_1||$ and $||f_2||$.
(B) Compute $d(f_1, f_2)$.

We consider the function

$$g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

(C) Use the fixed point theorem to show that there exists a number $x^* \geq 1$ such that $g(x^*) = x^*$.