LECTURE 7

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DRE 7017

1. Differential equations
2. Systems of linear differential equations
3. Linear approximations

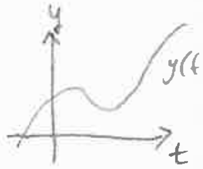
FMEA 5-7

ME 24-25

① DIFFERENTIAL EQUATIONS

Models with (time as) a continuous variable

- An ORDINARY DIFFERENTIAL EQUATION (ODE) is an equation in $t, y, y', y'', \dots$ where the unknown is the function $y = y(t)$.



- The ORDER of an ODE is the highest order derivative that appears in the equation

- FIRST ORDER $\quad y' = F(y,t)$

  Separable: $y' = f(y) \cdot g(t)$ - Integration y and t

  First order linear : $y' + a(t)y = b(t)$ - Integrating factor $e^{\int a(t)dt}$

  Autonomous : $y' = F(y)$ - Independent of time

- FIRST ORDER LINEAR AUTONOMOUS ODE :

$$y' = F(y) = ay + b \quad , \quad a, b \in \mathbb{R}, \quad a \neq 0$$

Is actually separable ... so the usual way of solving it is ...

$$\frac{1}{a}\ln(ay+b) = \int \frac{1}{ay+b}\, dy = \int 1\, dt = t + C$$

$$\ln(ay+b) = at + C$$

$$ay + b = e^{at+C}$$

$$ay = Ce^{at} - b$$

$$y(t) = Ce^{at} - \frac{b}{a}$$

If $b = 0$: homogeneous

If $b \neq 0$: nonhomogeneous

ALT: consider 1st order linear and integrating factor

$$y' - ay = b \qquad f(t) = -a$$

$$(e^{-at}y)' = be^{-at} \quad \wr \int dt \qquad F(t) = -at$$

$$e^{-at}y = -\frac{b}{a}e^{-at} + C \qquad e^{F(t)} = e^{-at}$$

$$y(t) = Ce^{at} - \frac{b}{a}$$

But can do better...

STEADY STATE (equilibrium state, stationary solution, rest point)

$F(y) = 0$: $\quad ay + b = 0$

$y_e = -\frac{b}{a}$ ← Note that this is a constant

CHANGE OF VARIABLES $\quad z = y - y_e$

then $\quad z' = y'$

and $\quad y' = ay + b$, so $z' = a(z + y_e) + b$

$$= az + \underbrace{ay_e + b}_{= 0}$$

reduces to $\quad z' = az \quad$ with $\quad z_e = 0$

and obvious solution

$$\underline{z = Ce^{at}}$$

since $\quad z' = Cae^{at} = az$.

(or use $z' - az = 0$ and char eq: $r - a = 0$, $r = a$.

GENERAL SOLUTION

So $\quad z + y_e = \underline{\underline{y = Ce^{at} - \frac{b}{a}}}$

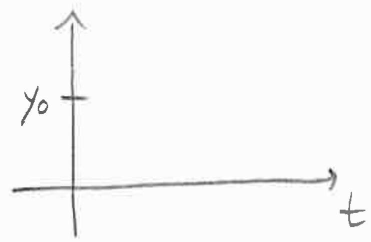
PARTICULAR SOLUTION (with INITIAL CONDITION)

Want $\quad y(t) = Ce^{at} - \frac{b}{a} \quad$ to hold with $y(0) = y_0$

$(y(c) = y_c)$

$$y_0 = y(0) = Ce^{a \cdot 0} - \frac{b}{a}$$

$$= C - \frac{b}{a}$$



so $\quad \underline{C = y_0 + \frac{b}{a}}$

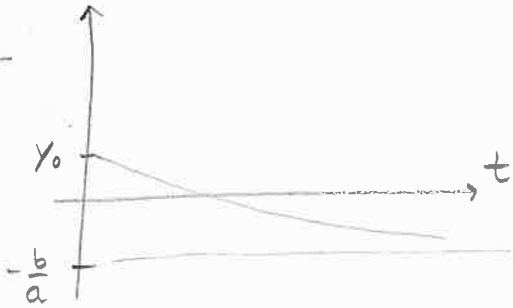
The particular solution is $\quad \underline{\underline{y(t) = (y_0 + \frac{b}{a})e^{at} - \frac{b}{a}}}$

DIFFERENT STEADY STATES

$$y(t) = \left(y_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}$$

STABLE: $a<0$

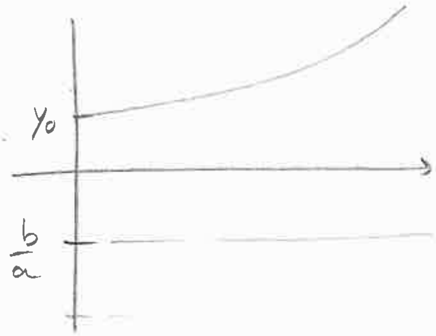
If y_0 is close to (but not equal) y_e, then $\lim_{t\to\infty} y(t) = y_e$.



$a<0$
$\Rightarrow e^{at} \to 0$

UNSTABLE $a>0$

If y_0 is close to (but not equal) y_e, then $y(t)$ would move away from y_e.



$a>0$
$\Rightarrow e^{at} \to \infty$
$(\text{or } -\infty)$
dep on y_0 and $\frac{b}{a}$

(2) SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

(FIRST ORDER)

$$y_1' = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n + b_1$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n + b_2$$
$$\vdots$$
$$y_n' = a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n + b_n$$

$$\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad A = (a_{ij})$$

$$\underline{y}' = A\underline{y} + \underline{b}$$

STEADY STATE:

$$\underline{y}' = \underline{0}, \text{ so } A\underline{y} + \underline{b} = \underline{0}$$

$\underline{y_e}$ is the solution of $A\underline{y} = -\underline{b}$

Vector in $\mathbb{R}^n$ of constants

Recall how the solutions change depending on det A and if $\underline{b} = \underline{0}$!

CHANGE OF VARIABLES

$$\underline{z} = \underline{y} - \underline{y_e}$$

$$\begin{aligned}
\underline{z}' = \underline{y}' &= A\underline{y} + \underline{b} \\
&= A(\underline{z} + \underline{y_e}) + \underline{b} \\
&= A\underline{z} + \underbrace{A\underline{y_e} + \underline{b}}_{= \underline{0}} \\
&= A\underline{z}
\end{aligned}$$

Slightly harder to solve than in the simple case.

SOLUTION VIA DIAGONALIZATION

Holds if ① A has n distinct eigenvalues

or ② $\dim E_{\lambda_i} = m_i$ for all distinct eigenvalues λ_i.

the multiplicity of the eigenvalue λ_i

The eigenspace of λ_i spanned by the eigenvectors associated to λ_i.

Then $A = PDP^{-1}$ for $P = (\underline{v}_1 | \dots | \underline{v}_n)$

$P^{-1}AP = D$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

← The orders of the $\underline{v}_i$ and λ_i are important

- 2ND CHANGE OF VARIABLES

$P\underline{u} = \underline{z}$ $\qquad (P\underline{u})' = \underline{z}' = A\underline{z}$

$$P\underline{u}' = AP\underline{u} \qquad \Big| \cdot P^{-1} \rightarrow$$

$$\underline{u}' = \underbrace{P^{-1}AP}\underline{u}$$

$$* \quad \underline{u}' = D\underline{u}$$

Then the solution is immediate from the simple case

$$* \left.\begin{aligned} u_1' &= \lambda_1 u_1 \\ u_2' &= \lambda_2 u_2 \\ &\vdots \\ u_n' &= \lambda_n u_n \end{aligned}\right\} \Rightarrow \begin{aligned} u_1 &= C_1 e^{\lambda_1 t} \\ u_2 &= C_2 e^{\lambda_2 t} \\ &\vdots \\ u_n &= C_n e^{\lambda_n t} \end{aligned}$$

BACKWARDS SUBSTITUTION

① $\underline{z} = P \cdot \underline{u}$

$$= (\underline{v}_1 | \dots | v_n) \cdot \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$= c_1 e^{\lambda_1 t} \underline{v}_1 + \dots + c_n e^{\lambda_n t} \underline{v}_n$$

② $\underline{y} = \underline{z} + \underline{y}_e$

$$= c_1 e^{\lambda_1 t} \underline{v}_1 + \dots + c_n e^{\lambda_n t} \underline{v}_n + \underline{y}_e$$

EX: $y_1' = y_1 - 3y_2 + 2$

$y_2' = 2y_1 - 4y_2 + 2$

$$\underline{y}' = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix} \cdot \underline{y} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

<u>Steddy</u> $\underline{y}_e$ solves: $\left(\begin{array}{cc|c} 1 & -3 & -2 \\ 2 & -4 & -2 \end{array}\right)$

CRAMER'S RULE:

$$y_{e,1} = \frac{\begin{vmatrix} -2 & -3 \\ -2 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix}} = \frac{2}{2} = 1$$

$$y_{e,2} = \frac{\begin{vmatrix} 1 & -2 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix}} = \frac{2}{2} = 1$$

$$\underline{y}_e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>CoV 1</u>: $\underline{z}' = A\underline{z}$

$$= \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix} \underline{z}$$

$$\underline{z} = \underline{y} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>CoV 2</u>: Diagonalize A:

$$0 = \begin{pmatrix} 1-\lambda & -3 \\ 2 & -4-\lambda \end{pmatrix} = (1-\lambda)(-4-\lambda) + 6$$

$$= \lambda^2 + 3\lambda + 2 \quad \text{solve using } \lambda = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

(see directly...)

$$= (\lambda+1)(\lambda+2)$$

$\underline{\lambda_1 = -1} \wedge \underline{\lambda_2 = -2}$

<u>$\lambda_1 = -1$</u>: $\begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

<u>$\lambda_2 = -2$</u>: $\begin{pmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\left(\begin{aligned} &\text{so..} \quad P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \\ &P^{-1} = \frac{1}{3-2} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \end{aligned} \right)$$

we don't need this

GOING BACKWARDS

① $$\underline{z} = C_1 e^{-t} \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

② $$\underline{y} = \underline{z} + \underline{y}_e = C_1 e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

SOLUTION
$$\begin{cases} y_1 = 3C_1 e^{-t} + C_2 e^{-2t} + 1 \\ y_2 = 2C_1 e^{-t} + C_2 e^{-2t} + 1 \end{cases}$$

IMPORTANT OBSERVATION

A linear differential equation of order n can be rewritten to a system of first order linear differential equations.

$$y^{(n)} + c_1 y^{(n-1)} + \dots + c_{n-1} y' + c_n y = d$$

EX:

$$y^{(3)} + 2y'' + 3y' + 4y = 5$$

$$\left.\begin{aligned} y &= y \\ y' &= z \\ z' = y'' &= w \\ w' = y^{(3)} &= 5 - 4y - 3z - 2w \end{aligned}\right\}$$

$$\begin{pmatrix} y \\ z \\ w \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{pmatrix} \begin{pmatrix} y \\ z \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

STEADY STATES AND THEIR STABILITY

GLOBALLY ASYMPTOTICALLY STABLE:
A steady state solution y_e of $y' = F(y)$ is GLOBALLY ASYMPTOTICALLY STABLE if for any initial condition y_0, the solution of the initial value problem $y' = F(y)$, $y(0) = y_0$ tends to y_e as $t \to \infty$.

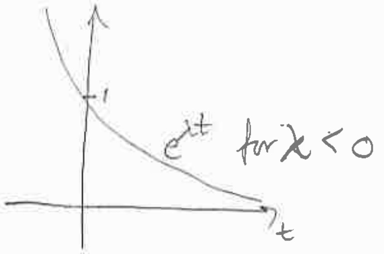
ASYMPTOTICALLY STABLE EQUILIBRIUM
A steady state y_e is an ASYMPTOTICALLY STABLE EQUILIBRIUM (of $y' = F(y)$) if every solution $y(t)$ which starts near y_e converges to y_e as $t \to \infty$.

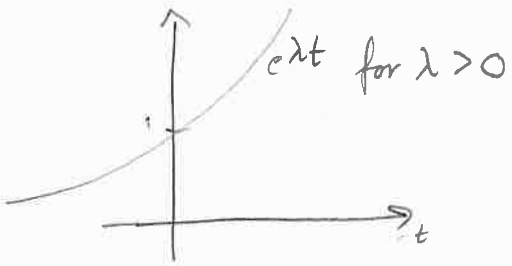
NEUTRALLY STABLE
A steady state y_e of $y' = F(y)$ is called NEUTRALLY STABLE if it is not locally asymptotically stable and if all solutions which start close enough to y_e stay close to y_e as $t \to \infty$.

STABLE if y_e asymptotically or neutrally stable
otherwise UNSTABLE

$$\underline{y}(t) = C_1 \underline{v}_1 e^{\lambda_1 t} + \dots + C_n \underline{v}_n e^{\lambda_n t} + \underline{y}_e$$

A term $C_i \underline{v}_i e^{\lambda_i t} \to \underline{0} \iff \lambda_i < 0$.





Recall: $n = 2$

$$\begin{cases} tr\, A = \lambda_1 + \lambda_2 \\ det\, A = \lambda_1 \cdot \lambda_2 \end{cases} \quad \text{and thus} \quad \begin{cases} \lambda_1, \lambda_2 < 0 \\ \iff \\ tr\, A < 0 \\ det\, A > 0 \end{cases}$$

WHAT IF THE EIGENVALUES ARE COMPLEX NUMBERS?

$z = a + ib$, $i^2 = -1$, $a, b \in \mathbb{R}$

a: real part, b: imaginary part

- The characteristic polynomial $det(A - \lambda \underline{I}) = 0$ is a polynomial of degree n in λ.
- By the fundamental theorem of algebra it has exactly n solutions over the complex numbers $\mathbb{C}$,
- If $det(A - \lambda I)$ has only real coefficients, then any complex solution $r = a + bi$ comes with a solution $\bar{r} = a - ib$, (its conjugate).
- $e^{(a \pm ib)t} = e^{at} e^{\pm ibt} \overset{\text{Euler}}{=} e^{at} (\underbrace{\cos \beta t + i \sin \beta t}_{|\ \ | < 1})$
- $\lim_{t \to \infty} e^{(a \pm ib)t} \to 0 \iff a < 0$

THM (ME 25.4)

a) If every real eigenvalue of A is negative and every complex eigenvalue of A has negative real part, then y_e is a globally asymptotically stable steady state of $y' = F(y) = Ay + \underline{b}$.

b) If A has a positive real eigenvalue or a complex eigenvalue with positive real part, then y_e is an unstable steady state.

c) If A has a zero eigenvalue or a purely imaginary eigenvalue that does <u>not</u> have a complete set of indep. eigenvectors, then y_e is an unstable rest point.

d) If A has a real eigenvalue equal to zero or a complex eigenvalue that is purely imaginary, if all such eigenvalues have a complete set of indep. eigenvectors, and if all the other eigenvalues have negative real part, then y_e is a neutrally stable steady state.

MORE IMPORTANTLY:

Look at the solutions and the coefficients and the terms!!

EX: $\underline{y}' = \begin{pmatrix} -2 & 5 \\ -1 & 0 \end{pmatrix} \underline{y}$, $\underline{y}_e = \underline{0}$

$\sqrt{-16} = 4\sqrt{-1}$

$0 = \det \begin{pmatrix} -2-\lambda & 5 \\ -1 & -\lambda \end{pmatrix}$

$= \underline{\lambda^2 + 2\lambda + 5}$

$\lambda = \dfrac{-(2) \pm \sqrt{2^2 - 4\cdot 1\cdot 5}}{2}$

$\underline{\lambda = -1 \pm 2i}$

Don't need to compute $\underline{v}_i$ or the solution to determine that (since λ_i has negative real part), $\underline{y}_e = \underline{0}$ is a globally asymptotically stable steady state.

3 LINEAR APPROXIMATIONS

$$\left.\begin{aligned} y_1' &= F(y_1, y_2) \\ y_2' &= G(y_1, y_2) \end{aligned}\right\}$$ A system of differential equations where F, G are general (nonlinear) functions.

STEADY STATE

$$\underline{y} = \underline{y_e} \quad \text{s.t.} \quad F(\underline{y_e}) = G(\underline{y_e}) = 0$$

LINEARIZATION

$$y_1' = F_{y_1}'(\underline{y_e})(y_1 - y_{e,1}) + F_{y_2}'(\underline{y_e})(y_2 - y_{e,2})$$

$$y_2' = G_{y_1}'(\underline{y_e})(y_1 - y_{e,1}) + G_{y_2}'(\underline{y_e})(y_2 - y_{e,2})$$

Replace functions with linear approximations at the steady state.

$\overset{\parallel}{\text{"tangents"}}$

Compact notation:

$$\underline{y}' = A(\underline{y} - \underline{y_e}) \quad \Longleftrightarrow \quad \underline{z}' = A\underline{z} \qquad \underline{z} = (\underline{y} - \underline{y_e})$$

$$A = \begin{pmatrix} F_{y_1}'(\underline{y_e}) & F_{y_2}'(\underline{y_e}) \\ G_{y_1}'(\underline{y_e}) & G_{y_2}'(\underline{y_e}) \end{pmatrix}$$

EX:
$$x' = x - 3y + 2x^2 + y^2 - xy$$
$$y' = 2x - y - e^{x+y} + 1$$

One steady state is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$F_x' = 1 + 4x - y \qquad F_y' = -3 + 2y - x$

$F_x'(\underline{0}) = 1 \qquad F_y'(\underline{0}) = -3$

$G_x' = 2 - e^{x+y} \qquad G_y' = -1 - e^{x+y}$

$G_x'(\underline{0}) = 1 \qquad G_y'(\underline{0}) = -1$

$$A = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right).$$

$\det A = -2 + 3 > 0$

$\operatorname{tr} A = 1 - 2 = -1 < 0$

So globally asymp. st. at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.