LECTURE 10

28.09.2020

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DRE 7017

1. FIXED POINTS & FIXED POINT THEOREMS
   - Brouwer
   - Contractions
   - Bellman equation

2. CORRESPONDENCES
   - Kakutani

FIXED POINTS

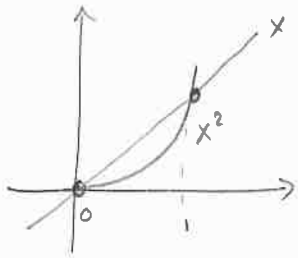
$f: X \to X$ operator $x \in X$ fixed point if $f(x) = x$

EX: $f(x) = x^2$

$x^2 = x$

$x(x-1) = 0$

$\underline{x=0} \wedge \underline{x=1}$



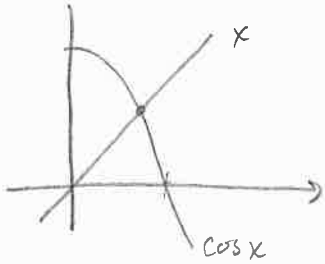
$f: [0,1] \longrightarrow [0,1]$

two fixed points

EX $f(x) = \cos x$

$\cos x = x$

Hard to find, but exists.

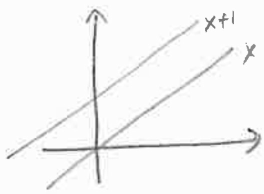


$f: [0, \frac{\pi}{2}] \longrightarrow [0, \frac{\pi}{2}]$

one fixed point.

EX $f(x) = x+1$

$x + 1 = x$



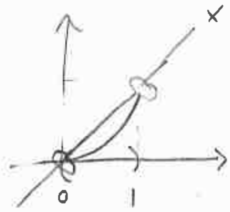
No fixed point

$f: \mathbb{R} \to \mathbb{R}$

(not bounded)

EX $f(x) = x^2$

$x^2 = x$



$f: (0,1) \longrightarrow (0,1)$

$(0,1)$ not closed

EX A nxn-matrix $A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\underline{x} \longmapsto A\underline{x}$

Fixed points are solutions to:

$A\underline{x} = \underline{x}$, so eigenvector corresponding to $\lambda = 1$,

(Equilibrium in Markov Chains)

BROUWER'S FIXED POINT THM

$$\left.\begin{array}{l} K \text{ non-empty, compact, convex} \\ f: K \to K \text{ continuous} \end{array}\right\} f: K \to K \text{ has a fixed point}$$

- "You can't change everything at once in a continuous way."
- Only existence
- At least one

CONTRACTIONS – functions that "shrinks"

(X,d) metric space

$f: X \to X$ is called

a CONTRACTION if there is a β with $0<\beta<1$
s.t.

$$d(f(x), f(y)) \leq \beta \cdot d(x,y) \quad (\text{i.e. } d(f(x), f(y)) < d(x,y))$$

for all $x, y \in X$.

EX: $f(x) = \frac{1}{2}x$ $\quad f: \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} d(f(x), f(y)) &= d(\tfrac{1}{2}x, \tfrac{1}{2}y) \\ &= |\tfrac{1}{2}x - \tfrac{1}{2}y| \\ &= \tfrac{1}{2}|x-y| \\ &= \tfrac{1}{2}d(x,y), \text{ so a contraction with } \beta = \tfrac{1}{2} \end{aligned}$$

FACT: All contractions are continuous.
Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{\beta}$.

FIXED POINT THM FOR CONTRACTIONS

If $X = (X,d)$ is a non-empty, complete metric space and $f: X \to X$ is a contraction, then f has a unique fixed point.

PROOF BY CONSTRUCTION

Construct a sequence in X by
$x_0 \in X$
$x_1 = f(x_0)$
$x_2 = f(x_1) = f^2(x_0)$
$\vdots$

Need to show that this converges to $x \in X$, but since X is assumed to be complete, it is enough to prove that $\{x_i\}$ is a Cauchy sequence.

Observe that

$$d(x_1,x_2) = d(f(x_0),f(x_1)) \leq \beta d(x_0,x_1)$$

$$d(x_2,x_3) = d(f(x_1),f(x_2)) \leq \beta d(x_1,x_2) \leq \beta^2 d(x_0,x_1)$$

$$\vdots$$

$$d(x_n,x_{n+1}) \leq \beta^n d(x_0,x_1)$$

Let $m>n$ and consider

$$d(x_n,x_m) \overset{\triangle}{\leq} d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \dots + d(x_{m-1},x_m)$$

$$\leq \beta^n d(x_0,x_1) + \beta^{n+1} d(x_0,x_1) + \dots + \beta^{m-1} d(x_0,x_1)$$

$$= d(x_0,x_1) \sum_{k=n}^{m-1} \beta^k$$

$$= d(x_0,x_1)\beta^n \cdot \sum_{k=0}^{m-1-n} \beta^k$$

$$= d(x_0,x_1)\beta^n \cdot \frac{(1-\beta^{m-1-n})}{1-\beta} \quad \leq 1$$

$$\leq \underset{\text{Fixed}}{d(x_0,x_1)} \cdot \underset{\to 0}{\frac{\beta^n}{1-\beta}}, \text{ so Cauchy}$$

Choose N s.t. $\frac{\beta^N}{1-\beta} d(x_0,x_1) < \varepsilon$, then so is $\frac{\beta^n}{1-\beta} d(x_0,x_1)$ for all $m, n \geq N$

Since X complete $\{x_n\} \to x^*$

Since f is continuous

$$f(x^*) = f(\lim_{n\to\infty} x_n) \overset{\downarrow}{=} \lim_{n\to\infty} (f(x_n)) \left(= \lim_{n\to\infty} f^{n+1}(x_0)\right)$$

$$= \lim_{n\to\infty} x_{n+1}$$

$$= x^*,$$

so x^* is a fixed point.

- Uniqueness: Assume y^* is another fixed point: $x^* \neq y^*$. Then $d(x^*,y^*) > 0$

Since x^* and y^* are fixed points, and f a contraction,

$$d(x^*,y^*) = d(f(x^*),f(y^*)) < d(x^*,y^*)$$

gives a contradiction. $0<\beta<1$

BACK TO THE BELLMAN EQUATION (INFINITE HORIZON)

$$\mathcal{J}(x) = J_0(x) = \max_{u_t} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \text{ with } \begin{array}{l} x_0 \text{ given} \\ x_{t+1} = g(x_t, u_t) \\ u_t \in U \subset \mathbb{R} \end{array}$$

Bellman equation $\quad \mathcal{J}(x) = \max_{u \in U} \{ f(x,u) + \beta \mathcal{J}(g(x,u)) \}$

f bounded and $0 < \beta < 1 \Rightarrow \mathcal{J}(x)$ bounded

f, g continuous $\Rightarrow \mathcal{J}(x)$ continuous

so $\mathcal{J}(x) \in \mathcal{B}(\mathbb{R}, \mathbb{R})$ (vector space of bounded and cont. realvalued functions on $\mathbb{R}$,

SET UP

$$T : \mathcal{B}(\mathbb{R}, \mathbb{R}) \longrightarrow \mathcal{B}(\mathbb{R}, \mathbb{R})$$

$$\mathcal{J} \longmapsto \left(x \longmapsto \max_{u \in U} \{ f(x,u) + \beta \mathcal{J}(g(x,u)) \} \right)$$

bounded

CLAIM (1) T is a contraction (sup-norm)

Have: $d(\mathcal{J}_1, \mathcal{J}_2) = \sup |\mathcal{J}_1 - \mathcal{J}_2|$

and $T(\mathcal{J}_1)(x) = \sup \left[f(x,u) + \beta \mathcal{J}_1(g(x,u)) + \underbrace{\beta \mathcal{J}_2(g(x,u)) - \beta \mathcal{J}_2(g(x,u))}_{+0} \right]$

Reordering terms

$$= \sup \left[f(x,u) + \beta \mathcal{J}_2(g(x,u)) + \beta \left(\mathcal{J}_1(g(x,u)) - \mathcal{J}_2(g(x,u)) \right) \right]$$

$$\leq T(\mathcal{J}_2)(x) + \beta d(\mathcal{J}_1, \mathcal{J}_2),$$

and conversely, so

$$d(T(\mathcal{J}_1), T(\mathcal{J}_2)) = \sup |T(\mathcal{J}_1)(x) - T(\mathcal{J}_2)(x)| \leq \beta d(\mathcal{J}_1, \mathcal{J}_2)$$

CLAIM 2: $X = B(\mathbb{R}, \mathbb{R})$ is complete (with sup-norm).

Need to show that every Cauchy sequence in X is convergent.

Establish f

- Let $\{f_n\}$ be a Cauchy sequence in X: Given $\varepsilon > 0$, $\exists\ N > 0$ s.t.

$$\|f_n - f_m\|_\infty < \varepsilon \quad \forall\ n, m \geq N.$$

Since sup-norm, we have for all $x \in \mathbb{R}$

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon,$$

So $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$; but $\mathbb{R}$ is complete (Earlier lecture), so $f_n(x) \to f(x)$, and we have a function $f: X \to X$.

Bounded f

- A Cauchy sequence in a normed space must be bounded, so there is $M > 0$ s.t. $\|f_n\|_\infty \leq M$.

Since sup-norm, we have for all $x \in \mathbb{R}$

$$|f_n(x)| \leq \|f_n\|_\infty \leq M,$$

so $M \geq \lim_{n \to \infty} |f_n(x)| = |f(x)|$, and $f(x)$ is bounded, so $f(x) \in X$.

f as limit

- For each $x \in \mathbb{R}$ and $n \geq N$, take

$$|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| < \varepsilon$$

So for $n \geq N$ $\|f_n - f\|_\infty < \varepsilon$, and f is the limit of $\{f_n\}$ in $X = B(\mathbb{R}, \mathbb{R})$.

CONCLUSION: T has a unique fixed point

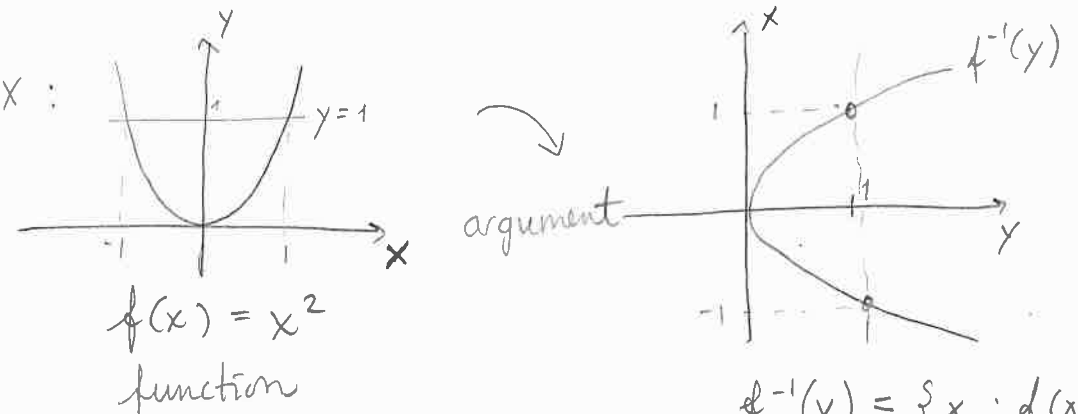
$$T(J^*) = J^*.$$

CORRESPONDENCES

- A CORRESPONDENCE $F: A \twoheadrightarrow B$ $\quad x \mapsto\!\!\!\twoheadrightarrow F(x)$
  is a map that associates with each element $x \in A$ a non-empty subset $F(x) \subseteq B$.
  In other words: A multivalued function.

- GRAPH of F: $\Gamma(F) = \{(x,y) \in A \times B : x \in A, y \in F(x)\}$
  (see PS. 10-2)

- EX:



$f^{-1}(y) = \{x : f(x) = y\}$
pre-image is a correspondence

- CONTINUITY comes in several different forms!
- One form is needed in a fixed point theorem...
- F is UPPER HEMICONTINUOUS (u.h.c) at $x \in A$ if for every open set U that contains $F(x)$ there exists a neighbourhood N of x such that $F(x) \subseteq U$ for every x in $N \cap A$, i.e. $F(N \cap A) \subseteq U$.

F is u.h.c iff u.h.c at every x in A.

All this very similar to continuity of functions

TEST (14.1.3 FMEA) - COMPACT GRAPH FOR U.H.C.

If a correspondence $F: A \twoheadrightarrow B$ has a compact graph, then it is upper hemicontinuous.

KAKUTANI'S FIXED POINT THEOREM

Let K be a nonempty, compact, convex set in $\mathbb{R}^n$ and F a correspondence $K \twoheadrightarrow K$.

Suppose that a) $F(x)$ is nonempty, convex set in K for each $x \in K$.

b) F is upper hemicontinuous.

Then F has a fixed point x^* in K,

$$x^* \in F(x^*).$$

NOTE: Kakutani (correspondences) is a generalization of Brouwer (functions)