Problem Sheet 8
DRE 7007 Mathematics

BI Norwegian Business School

Solutions Problem Set 8

1. $\max \int_0^2 (3 - x^2 - u^2)\, dt$ when $\begin{cases} x' = u \\ x(0) = 1 \\ x(2) = 4 \\ U = \mathbb{R} \end{cases}$

Hamiltonian: $H = p_0 (3 - x^2 - u^2) + pu$

$p_0 = 1$: $H = 3 - x^2 - u^2 + pu$

A) $H'_u = -2u + p = 0 \Rightarrow u = p/2$, $H''_{uu} = -2 < 0 \Rightarrow u = p/2$ is max.

B) $p' = -H'_x = 2x \Rightarrow u' = p'/2 = 2x/2 = x$

$$\left.\begin{aligned} u' &= x \\ x' &= u \end{aligned}\right\} \quad \begin{pmatrix} u \\ x \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ x \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \lambda_1 = 1, \lambda_2 = -1 \text{ eigenval.}$$

$\lambda_1 = 1$: $t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvect.

$\lambda_2 = -1$: $t \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ — " —

$$\Downarrow$$

$$\begin{pmatrix} u \\ x \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \begin{pmatrix} c_1 e^t + c_2 e^{-t} \\ c_1 e^t - c_2 e^{-t} \end{pmatrix}$$

$$u = C_1 e^t + C_2 e^{-t}$$
$$x = C_1 e^t - C_2 e^{-t}$$

$x(0) = 1$: $C_1 e^0 - C_2 e^0 = C_1 - C_2 = 1 \Rightarrow C_1 = 1 + C_2$

$x(2) = 4$: $C_1 e^2 - C_2 e^{-2} = 4$

$$(1 + C_2) e^2 - C_2 e^{-2} = 4$$

$$c_2 \cdot (e^2 - e^{-2}) = 4 - e^2 \Rightarrow C_2 = \frac{4 - e^2}{e^2 - e^{-2}}$$

$$C_1 = \frac{\cancel{e^2} - e^{-2} + 4 - \cancel{e^2}}{e^2 - e^{-2}} = \frac{4 - e^{-2}}{e^2 - e^{-2}}$$

Solution: $x(t) = \dfrac{4 - e^{-2}}{e^2 - e^{-2}} e^t - \dfrac{4 - e^2}{e^2 - e^{-2}} e^{-t}$

$u(t) = \dfrac{4 - e^{-2}}{e^2 - e^{-2}} e^t + \dfrac{4 - e^2}{e^2 - e^{-2}} e^{-t}$

Since $U = \mathbb{R}$ is convex and $H = 3 - x^2 - u^2 + pu$ is concave in (p, u) this is the optimal pair.

($p_0 = 0$ gives no solution, but it is not really necessary to check since we have optimal pair).

2. max $\int_0^T \left(x - \frac{1}{2}u^2\right) dt$ when $\begin{cases} \dot{x} = u \\ x(0) = x_0 \\ U = \mathbb{R} \end{cases}$

Hamiltonian: $H = p_0 \left(x - \frac{1}{2}u^2\right) + pu$

$p_0 = 1$: $H'_u = -u + p = 0 \Rightarrow u = p$ $\quad H''_{uu} = -1 < 0 \Rightarrow$ max at $u = p$.

$p' = -H'_x = -1 \Rightarrow p(t) = p_0 - t$

$p(T) = 0 \Rightarrow p(T) = p_0 - T = 0 \Rightarrow p_0 = T \Rightarrow p = T - t$

$$\left.\begin{array}{l} \dot{x} = u = p = T - t \Rightarrow x = -\frac{1}{2}t^2 + Tt + C \\ \qquad x(0) = C = x_0 \end{array}\right\} \Rightarrow \begin{array}{l} x(t) = -\frac{1}{2}t^2 + Tt + x_0 \\ u(t) = T - t \end{array}$$

Since $x - \frac{1}{2}u^2 + pu$ is concave in (x, u), this is the optimal pair.

$$V(x_0, T) = \int_0^T x - \frac{1}{2}u^2 \, dt = \int_0^T -\frac{1}{2}t^2 + Tt + x_0 - \frac{1}{2}(T-t)^2 \, dt$$

$$= \left[-\frac{1}{6}t^3 + \frac{T}{2}t^2 + x_0 t + \frac{1}{6}(T-t)^3\right]_0^T = -\frac{T^3}{6} + \frac{T^3}{2} + x_0 T - \frac{1}{6}T^3$$

$$= x_0 T + \frac{1}{6}T^3$$