Problem Sheet 7
DRE 7007 Mathematics

BI Norwegian Business School

Solutions Problem Set 7

1. $\underline{x}' = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \underline{x} + \begin{pmatrix} -5 \\ 1 \end{pmatrix}$

Steady state: $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \underline{x} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & -2 & 5 \\ 1 & 4 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 6 & -6 \end{array}\right)$

$\bar{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \Leftarrow \begin{array}{l} y = -1 \\ x = 3 \end{array}$

$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$:

$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = 0$

$\lambda_1 = 2, \lambda_2 = 3$

$\lambda_1 = 2$: $\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \underline{x} = \underline{0}$ $\quad \underline{x} = t \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\lambda_2 = 3$: $\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \underline{x} = \underline{0}$ $\quad \underline{x} = t \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General solution:

$$\underline{x} = \bar{x} + c_1 \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_1 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$$

Since $e^{2t}, e^{3t} \to \infty$ as $t \to \infty$, we must set $c_1 = c_2 = 0$ in order for the system to have $(x, y) \to (\bar{x}, \bar{y}) = (3, -1)$ as $t \to \infty$.

Hence $x_0 = 3, y_0 = -1$ is the only choice.

2. $\underline{x}' = \begin{pmatrix} 1 & -2 & -6 \\ 2 & 5 & 6 \\ -2 & -2 & -3 \end{pmatrix} \underline{x} + \begin{pmatrix} 7 \\ -4 \\ 4 \end{pmatrix}$

Steady state: $\left(\begin{array}{ccc|c} 1 & -2 & -6 & -7 \\ 2 & 5 & 6 & +4 \\ -2 & -2 & -3 & -4 \end{array}\right) \begin{array}{l} -2 \\ 2 \end{array} \to \left(\begin{array}{ccc|c} 1 & -2 & -6 & -7 \\ 0 & 9 & 18 & 18 \\ 0 & -6 & -15 & -18 \end{array}\right) \begin{array}{l} 2/3 \quad :9 \end{array}$

$\to \left(\begin{array}{ccc|c} 1 & -2 & -6 & -7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -6 & -12 \end{array}\right)$ $\quad \begin{array}{l} \bar{x} = 1 \\ \bar{y} = -2 \\ \bar{z} = 2 \end{array}$ $\quad \bar{x} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & -2 & -6 \\ 2 & 5 & 6 \\ -2 & -2 & -3 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -2 & -6 \\ 2 & 5-\lambda & 6 \\ -2 & -2 & -3-\lambda \end{vmatrix} = (1-\lambda)\left[(5-\lambda)(-3-\lambda)+12\right] + 2\left(2(-3-\lambda)+12\right) - 6\left(-4+2(5-\lambda)\right)$$

$$= (1-\lambda)(5-\lambda)(-3-\lambda) + \cancel{12} - \cancel{12\lambda} + 4(-3-\lambda) + \cancel{24} + \cancel{24} - \cancel{60} + \cancel{12\lambda}$$

$$= (-3-\lambda)(\lambda^2 - 6\lambda + 5 + 4) = (-3-\lambda)(\lambda-3)^2 = 0$$

$$\underline{\lambda_1 = -3}, \ \underline{\lambda_2 = \lambda_3 = 3}$$

$\underline{\lambda_1 = -3}$: $\begin{pmatrix} +4 & -2 & -6 \\ 2 & 8 & 6 \\ -2 & -2 & 0 \end{pmatrix} \xrightarrow{:-2} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 8 & 6 \\ 4 & -2 & -6 \end{pmatrix} \begin{matrix} \\ -2 \\ -4 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 6 & 6 \\ 0 & -6 & -6 \end{pmatrix}$

$x = z$
$y = -z$
z free

$\underline{x} = t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\underline{v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$

$\underline{\lambda_2 = 3}$: $\begin{pmatrix} -2 & -2 & -6 \\ 2 & 2 & 6 \\ -2 & -2 & -6 \end{pmatrix} \begin{matrix} :-2 \\ \\ \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x = -y - 3z$
y free
z free

$\underline{x} = \underset{y}{s} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \underset{z}{t} \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$\underline{v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}, \ \underline{v_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}$

General solution:

$$\underline{x} = \bar{x} + \sum c_i \underline{v_i} e^{\lambda_i t} = \underline{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + c_1 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{3t} + c_3 \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} e^{3t}}$$

Since $e^{3t} \to \infty$, $e^{-3t} \to 0$ as $t \to \infty$, $\underline{x} \to \bar{x}$ when $\underline{c_2 = c_3 = 0}$.

Since $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + c_1 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+c_1 \\ -2-c_1 \\ 2+c_1 \end{pmatrix}$ when $c_2 = c_3 = 0$,

this corresponds to the line in $\mathbb{R}^3$ with parametric description

$$\underline{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + c_1 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$$

3. $\xi' = g - (g+m)\xi$

$$\dot{g} = \left(\frac{L^N}{a} - g\right)\left(\rho + m + g - \frac{1-\alpha}{\alpha}\left(\frac{L^N}{a} - g\right)\frac{1}{\xi}\right)$$

Steady state: $g - (g+m)\xi = 0 \Rightarrow \underline{\xi = \frac{g}{g+m}}$ (write g, ξ instead of $\bar{g}, \bar{\xi}$ for simplicity)

$$\rho + m + g - \frac{1-\alpha}{\alpha}\left(\frac{L^N}{a} - g\right)\frac{1}{\xi} = 0$$

$$\Rightarrow \underline{\frac{1}{\xi} = \frac{\rho + m + g}{\left(\frac{1-\alpha}{\alpha}\right)\cdot\left(\frac{L^N}{a} - g\right)}}$$

Linearization at steady state:

$$\begin{pmatrix} \xi' \\ g' \end{pmatrix} = A \cdot \begin{pmatrix} \xi - \bar{\xi} \\ g - \bar{g} \end{pmatrix}, \text{ where } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$a_{11} = \frac{\partial}{\partial \xi}\left(g - (g+m)\xi\right) = \underline{-(g+m)} < 0$$

$$a_{12} = \frac{\partial}{\partial g}\left(g - (g+m)\xi\right) = 1 - \xi = 1 - \frac{g}{g+m} = \underline{\frac{m}{g+m}} > 0$$

$$a_{21} = \frac{\partial}{\partial \xi}\left(\left(\frac{L^N}{a} - g\right)\cdot\left(\rho + m + g - \frac{1-\alpha}{\alpha}\left(\frac{L^N}{a} - g\right)\frac{1}{\xi}\right)\right)$$

$$= \left(\frac{L^N}{a} - g\right)\cdot\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{L^N}{a} - g\right)\cdot\frac{1}{\xi^2}$$

$$= \left(\frac{L^N}{a} - g\right)^2\left(\frac{1-\alpha}{\alpha}\right)\cdot\frac{(\rho + m + g)^2}{\left(\frac{1-\alpha}{\alpha}\right)^2\cdot\left(\frac{L^N}{a} - g\right)^2} = \underline{\frac{\alpha}{1-\alpha}(\rho + m + g)^2} > 0$$

$$a_{22} = \frac{\partial}{\partial g}\left(\left(\frac{L^N}{a} - g\right)\left(\rho + m + g - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{L^N}{a} - g\right)\frac{1}{\xi}\right)\right)$$

$$= -1\cdot\underbrace{\left(\rho + m + g - \frac{1-\alpha}{\alpha}\left(\frac{L^N}{a} - g\right)\frac{1}{\xi}\right)}_{=0} + \left(\frac{L^N}{a} - g\right)\cdot\left(1 - \frac{1-\alpha}{\alpha}\cdot(-1)\frac{1}{\xi}\right)$$

$$= \left(\frac{L^N}{a} - g\right)\cdot\left(1 + \frac{1-\alpha}{\alpha}\cdot\frac{\rho + m + g}{\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{L^N}{a} - g\right)}\right) = \left(\frac{L^N}{a} - g\right) + (\rho + m + g)$$

$$= (\rho + m + g)\cdot\left(1 + \frac{(L^N/a - g)}{\rho + m + g}\right) = \underline{(\rho + m + g)\cdot\left(1 + \frac{\alpha}{1-\alpha}\cdot\frac{g}{g+m}\right)} > 0$$

$|A| = a_{11}a_{22} - a_{12}a_{21} < 0$ since $a_{11} < 0,\ a_{22} > 0,\ a_{12}, a_{21} > 0$

Let λ_1, λ_2 be the two eigenvalues of A (possibly complex ones)
Then there are two possibilities:

1) λ_1, λ_2 are real $\Rightarrow$ $\lambda_1 \cdot \lambda_2 = |A| \Rightarrow \lambda_1 > 0, \lambda_2 < 0$ or $\lambda_1 < 0, \lambda_2 > 0$

2) $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ are complex with $\lambda_1 + \lambda_2$ real $\Rightarrow$

$$\lambda_1 \cdot \lambda_2 = |A|$$
$$\lambda_1 \cdot \lambda_2 = (a+bi)\cdot(a-bi) = a^2 - b^2 i^2 = a^2 + b^2 \geq 0$$

contradiction

<u>Conclusion</u>: There is one positive and one negative eigenvalue for A.

$\Downarrow$

<u>Not globally asymptotically</u> stable

(Solution of lin. system: $\underline{x} = \bar{\underline{x}} + c_1 \cdot \underline{v}_1 \cdot e^{\lambda_1 t} + c_2 \cdot \underline{v}_2 e^{\lambda_2 t}$)

$c_1 \cdot \underline{v}_1 \cdot e^{\lambda_1 t} \to \infty$ since $\lambda_1 > 0$

$c_2 \cdot \underline{v}_2 e^{\lambda_2 t} \to 0$ since $\lambda_2 < 0$