Problem Sheet 5
DRE 7007 Mathematics

BI Norwegian Business School

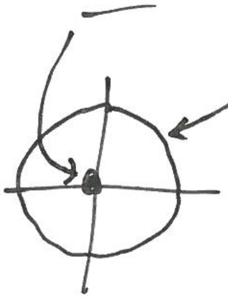
Solutions Problem Sheet 5

1. $f(x,y) = x^4 + 2x^2y^2 + y^4 - x^2 - y^2 = (x^2+y^2)^2 - (x^2+y^2) = u^2 - u$, $u = x^2+y^2$

$f'_x = 4x^3 + 4xy^2 - 2x = 2x(2x^2+2y^2-1) = 0 \Rightarrow x = 0$ or $x^2+y^2 = 1/2$

$f'_y = 4x^2y + 4y^3 - 2y = 2y(2x^2+2y^2-1) = 0 \Rightarrow y = 0$ or $x^2+y^2 = 1/2$

$\underline{(x,y) = (0,0)}$ or $\underline{x^2+y^2 = 1/2}$ (a circle)



$$D^2 f(x,y) = \begin{pmatrix} 12x^2+4y^2-2 & 8xy \\ 8xy & 4x^2+12y^2-2 \end{pmatrix}$$

$(x,y) = (0,0)$: $D^2 f = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ neg. defn. $\Rightarrow$ $\underline{(0,0) \text{ is local max}}$

$\underline{(x,y) \mid x^2+y^2 = 1/2}$:

$D_1 = 12x^2+4y^2-2 = 8x^2 + 4(x^2+y^2) - 2 = 8x^2 \geq 0$

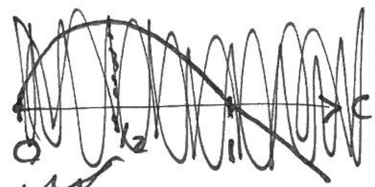
$\Delta_1 = 4x^2+12y^2-2 = 4(x^2+y^2) + 8y^2 - 2 = 8y^2 \geq 0$

$D_2 = 8x^2 \cdot 8y^2 - (8xy)^2 = 0$

pos. semidef. $\Downarrow$ no concl. in soc.

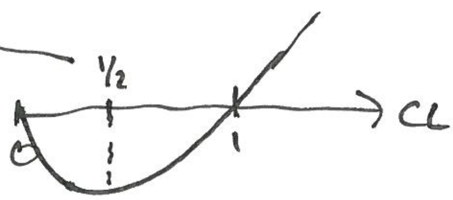
$f = u^2 - u$, $u = x^2+y^2$

If $u = c$, then $f = c^2 - c$ Profile:



$(u^2-u)' = 2u - 1 = 0$

$u = 1/2$

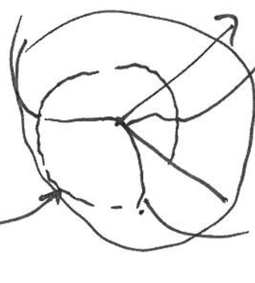


$u = 1/2 \Leftrightarrow x^2+y^2 = 1/2$ is $\underline{\text{local min.}}$

<u>Global min/max:</u>

No global max ($f \to \infty$ as $u \to \infty$)

<u>Global min:</u> $f = 1/4 - 1/2 = \underline{-1/4}$ at all (x,y) with $x^2+y^2 = 1/2$



2. $f(x,y) = \ln(1-x^2-y^2)$, $x^2+y^2<1$

$$f'_x = \frac{1}{1-x^2-y^2}\cdot(-2x) = \frac{-2x}{1-x^2-y^2}$$

$$f'_y = \frac{1}{1-x^2-y^2}\cdot(-2y) = \frac{-2y}{1-x^2-y^2}$$

$$f''_{xx} = \frac{-2(1-x^2-y^2)-(-2x)\cdot(-2x)}{(1-x^2-y^2)^2} = \frac{2y^2-2x^2-2}{(1-x^2-y^2)^2}$$

$$f''_{xy} = \frac{-(-2x)\cdot(-2y)}{(1-x^2-y^2)^2} = \frac{-4xy}{(1-x^2-y^2)^2}$$

$$f''_{yy} = \frac{2x^2-2y^2-2}{(1-x^2-y^2)^2} \qquad \text{(by symmetry)}$$

$$D_1 = \frac{2y^2-2x^2-2}{(1-x^2-y^2)^2} \leq \frac{2y^2+2x^2-2}{(1-x^2-y^2)^2} = \frac{-2}{1-x^2-y^2} < 0$$

$$D_2 = \frac{(2y^2-2x^2-2)(2x^2-2y^2-2)-16x^2y^2}{(1-x^2-y^2)^4} = \frac{-(2x^2-2y^2)^2+2^2-16x^2y^2}{(1-x^2-y^2)^4}$$

$$= \frac{-(2x^2+2y^2)^2+4}{(1-x^2-y^2)^4} = \frac{-4((x^2+y^2)^2-1)}{(1-x^2-y^2)^4} = \frac{4(1-(x^2+y^2)^2)}{(1-x^2-y^2)^4} > 0$$

$\Downarrow$

f is concave $\Rightarrow$ f is quasi-concave

Critical pts: $2x=2y=0 \Rightarrow (x,y)=(0,0)$ global max , $f(0,0)=\ln 1=0$

3. $f(x,y) = 3x^4 + 3x^2y - y^3$

$f'_x = 12x^3 + 6xy = 0$

$f'_y = 3x^2 - 3y^2 = 0 \Rightarrow x^2 - y^2 = 0 \Rightarrow x = \pm y$

$x = y$: $12x^3 + 6x^2 = 0$

$x^2 = 0, \quad 12x + 6 = 0$

$x = 0 \quad x = -1/2 \quad \Rightarrow (0,0), (-1/2, -1/2)$

$x = -y$: $12x^3 - 6x^2 = 0$

$6x^2 = 0 \quad 2x - 1 = 0$

$x = 0 \quad x = 1/2 \quad \Rightarrow (0,0), (1/2, -1/2)$

$$D^2 f(x,y) = \begin{pmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{pmatrix}$$

$$D^2 f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad D^2 f(-1/2, -1/2) = \begin{pmatrix} 6 & -3 \\ -3 & 3 \end{pmatrix} \quad \left.\begin{matrix} D_1 = 6 \\ D_2 = 9 \end{matrix}\right\} \text{pos. defn.}$$

$$D^2 f(1/2, -1/2) = \begin{pmatrix} 6 & 3 \\ 3 & 3 \end{pmatrix} \quad \left.\begin{matrix} D_1 = 6 \\ D_2 = 9 \end{matrix}\right\} \text{pos. defn.}$$

$(x,y) = (\pm 1/2, -1/2)$ $\quad f = \frac{3}{16} - \frac{3}{8} + \frac{1}{8} = \frac{3 - 2 \cdot 2}{16} = \frac{-1}{16}$ local min.

$(x,y) = (0,0)$: $\quad f = 0$ saddle point since $f(0,\varepsilon) = -\varepsilon^3 < 0$ when $\varepsilon > 0$

& $\quad > 0$ —"— $\varepsilon < 0$

No global max: $f(x,0) = 3x^4 \to \infty$ when $x \to \infty$

No global min; $f(0,y) = -y^3 \to -\infty$ when $y \to -\infty$