Problem Sheet 2
DRE 7007 Mathematics

BI Norwegian Business School

Problems

1. Let $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and let $T = \{(x,y) \in \mathbb{R}^2 : xy = 1\}$. Sketch the regions S and T in the plane, and find the boundaries ∂S and ∂T. For each of the regions, determine if it is open, closed, bounded, compact.

2. Consider the sequence given by $x_n = \frac{n^2+1}{n}$. Does the sequence have a limit? Is it bounded?

3. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the Euclidean norm. Show that the Cauchy-Schwarz inequality implies that the triangle inequality

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

4. Let $I = [0,1]$, and consider the vector space $C(I,\mathbb{R})$ of continuous functions on I. Show that

$$f \cdot g = \int_0^1 f(t)g(t)\mathrm{d}t$$

defines an inner product on $C(I,\mathbb{R})$, and compute $f \cdot g$ when $f(t) = t^2$ and $g(t) = t^3$. What is $d(f,g)$ when d is the metric induced by this inner product?

5. Let $I = [0,1]$, and consider the vector space $C(I,\mathbb{R})$ of continuous functions on I. The formula

$$\|f\| = \sup_{t \in [0,1]} |f(t)|$$

defines a norm on $C(I,\mathbb{R})$, called the *sup norm*. Use the sup norm to compute $d(f,g)$ when $f(t) = t^2$ and $g(t) = t^3$.

6 (Difficult). Let $I = [0,1]$, and let f_n be the function in $C(I,\mathbb{R})$ defined by $f_n = t^n$. When $C(I,\mathbb{R})$ has the sup norm, is the sequence (f_n) a Cauchy sequence? Hint: Compute $d(f_n, f_{n+1})$.

Keep answers as short and to the point as possible. Answers must be justified.