Plan:

1. Optimal control theory (discrete time)
2. Infinite horizon and the Bellman equation

Optimal control theory – current value formulation:

$$\max \int_{t_0}^{t_1} f(t,x,u)\, e^{-rt}\, dt \quad \text{when} \quad \begin{cases} x(t_0) = x_0 \\ x' = g(t,x,u) \\ u \in U \subseteq \mathbb{R} \\ x(t_1) \text{ free} \end{cases}$$

$$\underbrace{H = p_0 f(t,x,u)\, e^{-rt} + p \cdot g(t,x,u)}$$

$$H^c = \lambda_0 \cdot f(t,x,u) + \lambda \cdot g(t,x,u) \qquad \text{with} \quad \begin{array}{l} \lambda(t) = p(t) \cdot e^{rt} \\ \lambda_0 = p_0 \end{array}$$

current value Hamiltonian

Maximum principle

i) $u \mapsto H^c(t,x,u)$ has maximizer u^* ← means that $\frac{\partial H^c}{\partial u} = 0$ at interior pts.

ii) $\lambda' + r\lambda = -\partial H^c / \partial x$

iii) $\lambda(t_1) = 0$

Ex. in class:

$$\max \int_0^{20} (4K - u^2) e^{-0.25t}\, dt \quad \text{when} \quad \begin{cases} K(0) = K_0 \\ K' = -0.25K + u \\ u \in U = [0, \infty) \\ K(20) \text{ free} \end{cases}$$

Solution: using current value Hamiltonian

$$K(t) = 16 - \frac{16}{3} e^{0{,}5t - 10} + \left(K_0 - 16 + \frac{16}{3} e^{-10}\right) e^{-0.25t}$$

① Optimal control theory – discrete time & finite horizon

$$\max \sum_{t=0}^{T} f(t, x_t, u_t) \quad \text{when} \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(t, x_t, u_t) \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

Dynamic programming and Bellman's eqn.

$$s \le T: \quad J_s(x) = \max_{(u_s, \dots, u_T)} \sum_{t=s}^{T} f(t, x_t, u_t) \quad \text{when} \begin{cases} x_s = x \\ x_{t+1} = g(t, x_t, u_t) \\ u_t \in U \end{cases}$$

Bellman eqn:

$$\boxed{J_s(x) = \max_{u \in U} \{ f(s, x, u) + J_{s+1}(g(x, u)) \}} \quad (s < T)$$

Using this, we can determine $J_s(x)$ using an iterative process, starting with $s = T$ and working backwards.

Note:

$\underline{s = T}$: $\quad J_T(x) = \max_{u \in U} \{ f(T, x, u) \}$

Ex T:

$$\max \sum_{t=0}^{3} (1+x_t-u_t^2) \qquad x_{t+1}=x_t+u_t, \quad x_0=0, \quad U=\mathbb{R}$$

$$J_3(x) = \max_{u\in\mathbb{R}} (1+x-u^2) = 1+x \quad \text{at } u_3=0$$

$$J_2(x) = \max_{u_2\in\mathbb{R}} (1+x-u_2^2) + J_3(x+u_2) = \max_u \{1+x-u^2+1+x+u\}$$

$' = -2u+1 \qquad u=1/2$ ok.

$'' = -2$

$$= 2+2x+1/4 = \underline{9/4+2x}, \quad u_2 = 1/2$$

$$J_1(x) = \max \quad 1+x-u^2 + J_2(x+u) = \max \quad 1+x-u^2+9/4+2x+2u$$

$' = -2u+2 \qquad u=1$

$' = -2$

$$= 1+x-1+9/4+2x+2$$

$$= \underline{17/4+3x}, \quad u_1=1$$

$$J_0(x) = \max \quad 1+x-u^2 + 17/4 + 3(x+u)$$

$' = -2u+3=0 \qquad u=3/2$

$' = -2$

$$= 1+x-9/4+17/4+3x+9/2 = \underline{4x+15/2} \qquad u_0=3/2$$

With $x_0=0$:

$u_0 = 3/2$	$x_0 = 0$
$u_1 = 1$	$x_1 = 3/2$
$u_2 = 1/2$	$x_2 = 5/2$
$u_3 = 0$	$x_3 = 3$

$$J_0(x) = \underline{4x+15/2} = \underline{\underline{15/2}}$$

This will be explained in lecture 10.

(2) Infinite Horizon dynamic programming

$$\max \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \quad \text{subject to} \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(x_t, u_t) \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

Note:
* instead of $t \leq T$ we have infinite horizon $t \to \infty$
* $\beta^t f(x_t, u_t) = f(t, x_t, u_t)$; $g(x_t, u_t) = g(t, x_t, u_t)$
  ($0 < \beta < 1$ is the one-periode discount factor)
* we assume that $f(x_t, u_t)$ is bonded, i.e. $|f(x_t, u_t)| < M$ for all t for some number $M > 0$. This implies that the sum
$$\sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \leq \sum_{t=0}^{\infty} \beta^t M = \frac{M}{1-\beta} \quad \text{is finite}$$

Bellman equation: Let $J(x) = J_0(x) = \max\limits_{(u_t)} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t)$ subj to $\ldots \begin{cases} x_0 = x \\ \vdots \end{cases}$

$$J(x) = \max_{u \in U} \left\{ f(x, u) + \beta J(g(x, u)) \right\}$$

- Functional equations: we want to solve for the function $J(x)$. Difficult to find max when $J(x)$ is not known, hence difficult to find $J(x)$ by solving the Bellman equation.

- When $0 < \beta < 1$ and $|f(x_t, u_t)| < M$, the Bellman equation has a unique bounded solution $J^*(x)$. If we "guess" $\tilde{J}(x)$ and it fits in the equation, it is therefore the unique solution.

Ex:
Problem 12.3.1 in [FMEA]. See also SM Student Manual (online)