Plan:

1. Constrained optimization
2. Lagrange's method
3. Kuhn-Tucker formulation

1 Constrained optimization

Recall: $\max_{\underline{x} \in D} f(\underline{x})$ for $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$

constrained maximizer = maximizer $\underline{x}^* \in D$ s.t. $\underline{x}^* \in \partial D$ (boundary pt.)

Ex: $\min \underbrace{xyz}_{= f(x,y,z)}$ when $\begin{cases} x - y + 2z = 3 \\ x + y = 3 \end{cases}$

$D = \{(x,y,z) : \begin{matrix} x - y + 2z = 3 \\ x + y = 3 \end{matrix} \}$

$f : D \to \mathbb{R}$

$\partial D = D$

expected dimension of D is $\underset{\substack{\| \\ 3 \\ (\#\text{var's in } f)}}{n} - \underset{\substack{\| \\ 2 \\ (\#\text{constr.})}}{m} = 1$

2 Lagrange's method

Lagrange problems: Equality constraints

General form: max/min $f(x_1, \dots, x_n)$ when $\begin{cases} g_1(x_1, \dots, x_n) = a_1 \\ \vdots \qquad \vdots \\ g_m(x_1, \dots, x_n) = a_m \end{cases}$

Method: $\mathcal{L}(x_1, \dots, x_n; \lambda_1, \dots, \lambda_m) =$

$$f(x_1, \dots, x_n) - \lambda_1 \cdot g_1(x_1, \dots, x_n) - \dots - \lambda_m \cdot g_m(x_1, \dots, x_n)$$

$$\text{FOC} \begin{cases} \mathcal{L}'_{x_1} = 0 & f'_{x_1} - \lambda_1 (g_1)'_{x_1} - \dots - \lambda_m (g_m)'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 & \vdots \qquad \vdots \\ \vdots & \\ \mathcal{L}'_{x_n} = 0 & f'_{x_n} - \lambda_1 (g_1)'_{x_n} - \dots - \lambda_m (g_m)'_{x_n} = 0 \end{cases}$$

$$\text{C} \begin{cases} g_1(x_1, \dots, x_n) = a_1 \\ \vdots \\ g_m(x_1, \dots, x_n) = a_m \end{cases}$$

FOC + C : $n+m$ eqn's in $n+m$ unknowns

Solutions $\longrightarrow$ candidates for max/min.

NDCQ: non-degenerate constraint qualification

If $\underline{x}$ is a point in D $(g_1(\underline{x})=a_1, g_2(\underline{x})=a_2, \dots)$.

NDCQ at $\underline{x}$: $\operatorname{rk} J = m$

$$J = \begin{pmatrix} \partial g_1/\partial x_1(\underline{x}) & \partial g_1/\partial x_2(\underline{x}) & \dots & \partial g_1/\partial x_n(\underline{x}) \\ \vdots & & & \\ \partial g_m/\partial x_1(\underline{x}) & \partial g_m/\partial x_2(\underline{x}) & \dots & \partial g_m/\partial x_n(\underline{x}) \end{pmatrix}$$

$m \times n$ - matrix

$n - m$ = expected dim. of D

Thm: Given a Lagrange pb. in std. form, if $\underline{x}^*$ is a maximizer/minimizer and NDCQ is satisfied at $\underline{x}^*$, then there exist $(\lambda_1, \dots, \lambda_m)$ such that FOC + C are satisfied.

Solution method:
- Find all pts with FOC+C satisfied
- Find all pts. in D s.t. NDCQ is not satisfied.

If the Lagrange pb. has a solution, then it has to be one of these pts.

Sufficient conditions:

* If D is compact, then $\max/\min_{x \in D} f(\underline{x})$ has solutions.

$D = \{(x_1, x_2 \dots x_n) : g_1(x_1 \dots x_n) = a_1 \; \dots \; g_m(x_1 \dots x_n) = a_m\} \subseteq \mathbb{R}^n$

closed

D compact $\Leftrightarrow$ D bounded

* Second order conditions:

$$\boxed{\begin{array}{l} \text{If } (\underline{x}^*; \underline{\lambda}^*) \text{ satisfies FOC+C, then we have:} \\ \underline{x} \mapsto \mathcal{L}(\underline{x}; \underline{\lambda}^*) \text{ is concave} \Rightarrow \underline{x}^* \text{ is a maximizer} \\ \quad - \text{''} - \quad \text{convex} \Rightarrow \underline{x}^* \text{ is a minimizer} \end{array}}$$

Interpretation of Lagrange multiplier λ_i:

$f^*(a_i)$: optimal value fn., the max/min value as a fn. of a_i in the pb. $\max/\min f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Then: $\lambda_i = \dfrac{\partial f^*(a_i)}{\partial a_i}$

In particular: $\lambda_i > 0$ means that max/min would increase with increasing a_i.

③ Kuhn-Tucker formulation

Kuhn-Tucker problem in std. form: (inequality constr.)

$$\max f(x_1 \ldots x_n) \text{ when } \begin{cases} g_1(x_1,\ldots,x_n) \le a_1 \\ \vdots \qquad\qquad \vdots \\ g_m(x_1,\ldots,x_n) \le a_m \end{cases}$$

Kuhn-Tucker formulation:

$$\mathcal{L} = f(x_1,\ldots,x_n) - \lambda_1 \cdot g_1(\underline{x}) - \ldots - \lambda_m g_m(\underline{x})$$

complementary slackness conditions

$$\text{FOC} \begin{cases} \mathcal{L}'_{x_1} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{cases} \qquad C \begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases} \qquad \underline{\text{CSC}}: \begin{cases} \lambda_1 \ge 0 \text{ and} \\ \lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0 \\ \vdots \\ \lambda_m \ge 0 \text{ and} \\ \lambda_m \cdot (g_m(\underline{x}) - a_m) = 0 \end{cases}$$

Thm:

If ($\underline{x}^*$) is a maximizer in a Kuhn-Tucker problem in std form and NDCQ holds at $\underline{x}^*$, then there are $\lambda_1, \ldots, \lambda_m$ such that

$(\underline{x}^*, \underline{\lambda})$ satisfy FOC + C + CSC.

$\Downarrow$

1. All pts that satisfies FOC + C + CSC
2. All pts in D that does not satisfy NDCQ.

If the pb. has a maximizer, it is one of these pts in 1 or 2

~~Necc~~

Sufficient cond:

1. D compact $\Longrightarrow$ max/min $f(\underline{x})$ when $\underline{x} \in D$ has a solution

$\Uparrow$

D bounded

2. If $(\underline{x}^*, \lambda^*)$ satisfies FOC+C+CSC, and $\mathcal{L}(\underline{x}; \lambda^*)$ is concave (as a fn. of $\underline{x}$), then $\underline{x}^*$ is a maximizer. $\Big\}$ SOC

NDCQ in Kuhn-Tucker formulation:

Constraints: $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

For any pt $\underline{x} \in D$, certain constraints will be binding (hold by equality: $g_i(\underline{x}) = a_i$) and others will be non-binding (hold by strict inequality: $g_i(\underline{x}) < a_i$).

NDCQ: J' has maximal rank, where J' is obtained from the matrix $J = \left(\partial g_i / \partial x_j (\underline{x}) \right)$ by keeping rows corresponding to binding cond. at $\underline{x}$ and deleting — " — non-binding cond. at $\underline{x}$

Problem: max/min $\ln(x^2y) - x - y$ when $\begin{cases} x+y \geq 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$

a) ~~Draw the set D. Is it bounded?~~
Draw the set D. Is it bounded?

b) Does the problem have a max or a min? If so, find them.

Hint: You can use Kuhn-Tucker formulation, or divide up the problem. Which is easier?

You may apply the increasing fn. $\phi(x) = e^x$ to the objective function. Does this simplify the problem?