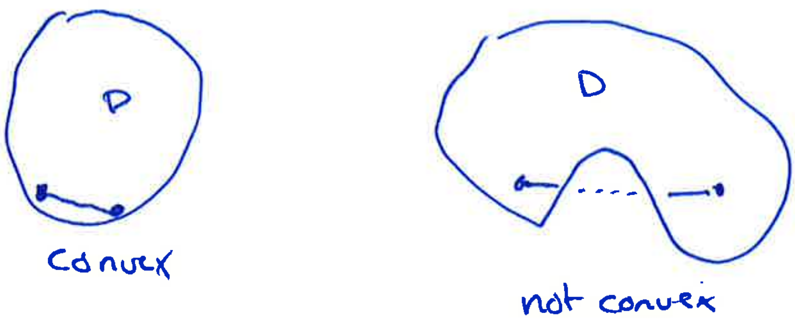
Plan:

1. Convex sets
2. Separation results
3. Convex / concave functions

1. Convex sets

A subset $D \subseteq \mathbb{R}^n$ is convex if $P, Q \in D \Rightarrow [P,Q] \subseteq D$, where $[P,Q] = \{\lambda \cdot P + (1-\lambda) \cdot Q \mid 0 \leq \lambda \leq 1\}$ is the line segment from P to Q.



Facts: $D, E \subseteq \mathbb{R}^n$ convex $\Rightarrow$ $D + E = \{P + Q \mid P \in D, Q \in E\}$ convex

$\{D_i : i \in I\}$ convex sets $\Rightarrow$ $\bigcap_I D_i$ is convex

Note: possible to define convex sets in inner product spaces (vector spaces) but not all metric spaces.

(2) Separation theorems

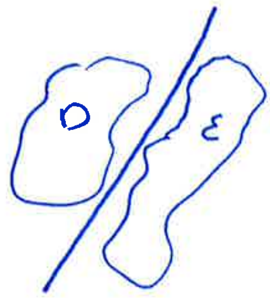
$H = \{\underline{x} \in \mathbb{R}^n \mid \underline{p} \cdot \underline{x} = a\} \subseteq \mathbb{R}^n$ is called a <u>hyperplane</u> when $\underline{p} \neq \underline{0}$ in $\mathbb{R}^n$, $a \in \mathbb{R}$.

$= \{\underline{x} : p_1 x_1 + \dots + p_n x_n = a\}$ (may assume $\|p\| = 1$).

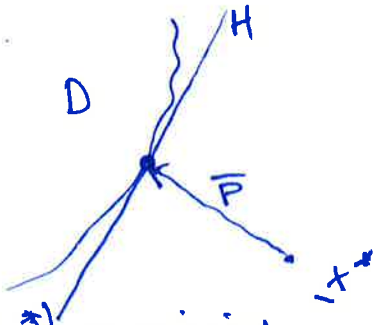
$D, \mathcal{E} \subseteq \mathbb{R}^n$ are <u>separated</u> by H if

$$D \subseteq \{\underline{x} \in \mathbb{R}^n : \underline{p} \cdot \underline{x} \geq a\}$$
$$\mathcal{E} \subseteq \{\underline{x} \in \mathbb{R}^n : \underline{p} \cdot \underline{x} \leq a\}$$

or conversely.



<u>Thm:</u> If $D, \mathcal{E} \subseteq \mathbb{R}^n$ are non-empty convex sets with $D \cap \mathcal{E} = \emptyset$, then there is a hyperplane seperating $D, \mathcal{E}$.



<u>Special case:</u> D closed, $\mathcal{E} = \{\underline{x}^*\}$ a pt.

Choose $y^* \in D$ s.t. $d(x^*, y^*)$ is minimal among all $y \in D$: This is possible.

<u>Put:</u> $\underline{p} = \underline{y}^* - \underline{x}^* \neq \underline{0}$

$H : \underline{p} \cdot \underline{x} = a$ with $a = \underline{p} \cdot \underline{y}$.

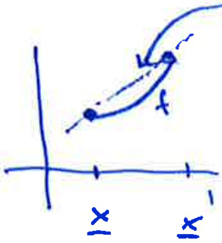
Then H separates $D, \mathcal{E}$.

③ Convex/concave functions

$f: D \to \mathbb{R}$ (for $D \subseteq \mathbb{R}^n$) is a fn. in n variables

Assume $D \subseteq \mathbb{R}^n$ convex. Then define:

f convex if



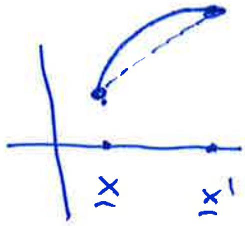
graph of f under (or on) line segment, or equivalently:

$f(\lambda \cdot \underline{x} + (1-\lambda)\underline{x}') \leq f(\underline{x}) \cdot \lambda + f(\underline{x}') \cdot (1-\lambda)$

for $0 \leq \lambda \leq 1$.

f strict convex if $<$ for $0 < \lambda < 1$.

f concave if



$\leftarrow f(\lambda \underline{x} + (1-\lambda)\underline{x}') \geq \lambda \cdot f(\underline{x}) + (1-\lambda) \cdot f(\underline{x}')$

f strict concave if $>$ for $0<\lambda<1$.

Alternative: f concave $\iff$ $-f$ convex

Result: ($D \subseteq \mathbb{R}^n$ open and convex set)

If $f: D \to \mathbb{R}$ is C^2, then:

$H(f)(\underline{x})$ pos. semidefin. for all $\underline{x}$ $\iff$ f convex

$H(f)(\underline{x})$ neg. — " — f concave

Since $H(f)(\underline{x})$ is symmetric matrix, this can be checked using

- eigenvalues, or
- principal minors

<u>Defn:</u> $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$ convex set

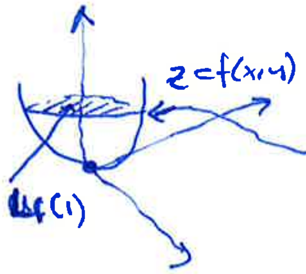
f is <u>quasi-concave</u> if $U_f(a) = \{x \in D : f(x) \geq a\}$ is a convex set for all a

" <u>quasi-convex</u> if $L_f(a) = \{x \in D : f(x) \leq a\}$ ~~is convex~~ is a convex set for all a.

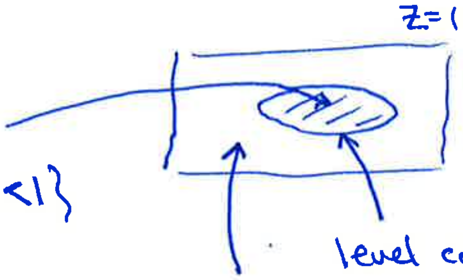
<u>Ex:</u> $f(x,y) = x^2 + y^2$ on $D = \mathbb{R}^2$

$H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is pos. defn

$\Rightarrow$ f is (strictly) <u>convex</u>



<u>Level curve:</u> $z = 1$



$L_f(1) =$
$\{(x,y) : x^2 + y^2 \leq 1\}$
the graph is <u>below</u> $z=1$
(L = lower)
convex set

level curve
$x^2 + y^2 = 1$

$U_f(1) =$
$\{(x,y) : x^2 + y^2 \geq 1\}$
the graph is <u>above</u> $z = 1$
(U = upper)
not convex set

By defn, this means that f is <u>quasi-convex</u>.

(since $L_f(a)$ is convex for all a, not just $a = 1$)

$$\boxed{\begin{array}{l} \underline{\text{Fact}}: \quad f \text{ convex} \Rightarrow f \text{ quasi-convex} \\ \qquad\quad\;\; f \text{ concave} \Rightarrow f \text{ quasi-concave.} \end{array}}$$