Plan:

1. Topology on metric spaces
2. Functions and continuity
3. Derivatives

Recall that (X,d) is a <u>metric space</u> means that X is a set, and $d : X \times X \to \mathbb{R}$ is a <u>metric</u> (distance function) on X
$(x,y) \mapsto d(x,y)$
satisfying

i) $d(x,y) \geq 0$ for all $x,y \in X$, and $d(x,y) = 0 \iff x = y$

ii) $d(x,y) = d(y,x)$ for all $x,y \in X$

iii) $d(x,y) + d(y,z) \geq d(x,z)$ for all $x,y,z \in X$.

1. <u>Topology</u> on a metric space (X,d)

<u>Open ball in X:</u> $B(p,M) = \{x \in X : d(x,p) < M\}$

<u>Closed ball in X:</u> $\overline{B}(p,M) = \{x \in X : d(x,p) \leq M\}$

<u>Defn:</u> $D \subseteq X$ is open if for all $x \in D$, there is an open ball $B(x,M) \subseteq D$.

∂D = boundary pts of X = all pts <u>$x \in X$</u> s.t. any open ball $B(x,M)$ contains pts in D <u>and</u> pts in D^c (= pts not in D)

D° = interior points in $D = D \setminus \partial D$
= all pts. in D that are <u>not</u> boundary pts.

$D \subseteq X$ is closed if D^c is open

Note: $D \cap \partial D = \phi \iff D$ is open

$\partial D \subseteq D \iff D$ is closed

D is bounded if there is an open ball $B(x, M)$ s.t. $D \subseteq B(x, M)$.

Defn: $K \subseteq X$ — K is compact if any sequence (x_i) in K has a subsequence $\{x_{i_1}, x_{i_2}, \ldots\}$ (with $i_1 < i_2 < \ldots$) that converges to a point in K.

Facts: K compact $\Rightarrow$ K closed and bounded

Bolzano-Weierstrass: $K \subseteq \mathbb{R}^n$ (Euclidean space) is compact $\iff$ K is closed and bounded.

2 Functions and continuity

$X=(X,d)$, $Y=(Y,d')$ } metric spaces

A function $f: X \to Y$ is a rule that assigns a value $f(x) \in Y$ to each point in X. X is called the domain and Y the codomain of the function.

$x \mapsto f(x) \in Y$

Ex: $f(x,y) = e^{xy}$, or

$$\boxed{\begin{aligned} f: \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\mapsto e^{xy} = f(x,y) \end{aligned}}$$

notation

$f(x,y) = \ln(x^2+y^2)$

$$\begin{aligned} f: D &\longrightarrow \mathbb{R} \\ (x,y) &\mapsto \ln(x^2+y^2) \end{aligned}$$

$D = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$

Ex: $$\left.\begin{aligned} T: \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}\right\} \text{ operators}$$

Ex: $$\left.\begin{aligned} \mathcal{J}: C(I,\mathbb{R}) &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_0^1 f(x)\,dx \end{aligned}\right\} \text{ functionals}$$

$C(I,\mathbb{R}) = \{ f: I = [0,1] \to \mathbb{R} \mid f \text{ is cont.} \}$

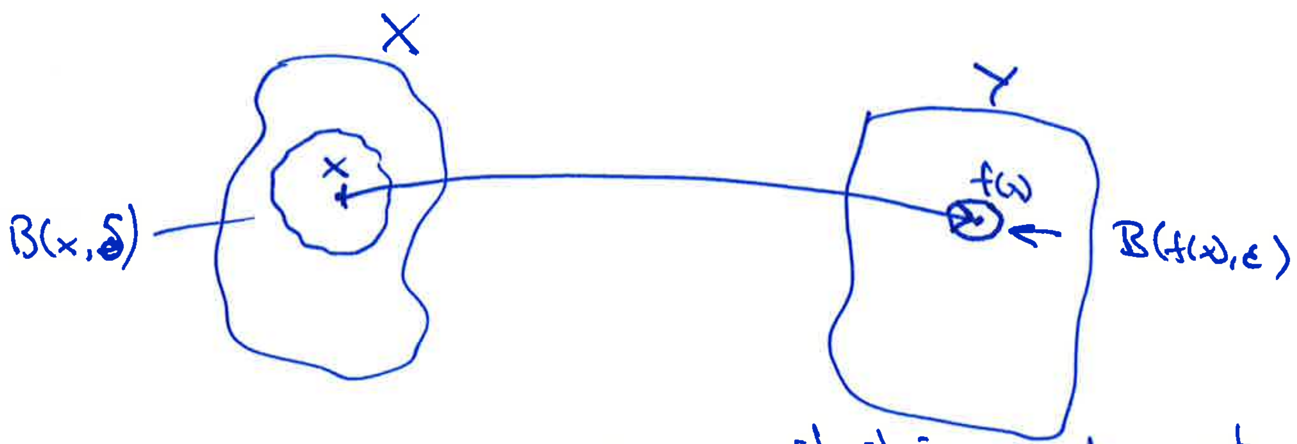
vector space, norm: $\|f\|_{\sup} = \sup_{x \in I} f(x)$

Let $f: X \to Y$ be a function. We say that f is <u>continuous</u> at $x \in X$ if the following condition holds:

For any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$x' \in B(x, \delta) \implies f(x') \in B(f(x), \varepsilon)$$

$$\left(d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon \right)$$



<u>Defn:</u> $f: X \to Y$ is continuous if it is cont. at all points $x \in X$.

<u>Proposition:</u>
If $f: X \to Y$ is cont. and $K \subseteq X$ is compact, then $\underset{\{f(x) : x \in K\}}{\underset{\shortparallel}{f(K)}} \subseteq Y$ is compact.

<u>Ex:</u> $f(x, y)$ cont. fn. defined on $D \subseteq \mathbb{R}^2$ closed and bounded $\implies$ $\max/\min_{(x,y) \in D} f(x, y)$ exists

<u>Why?</u> $f(D) \subseteq \mathbb{R}$ is compact.

Thm (Weierstrass)

If $D \subseteq \mathbb{R}^n$ is closed and bounded and $f : D \to \mathbb{R}$ is cont. then f attains a max/min. on D.

Facts:
- all "usual" functions on $\mathbb{R}^n$ are cont.

  $f(x,y) = e^{xy}$
- sums, differences, products, compositions of cont. functions are continuous.
- **Ex:**

  $$C(I,\mathbb{R}) \to \mathbb{R}$$
  $$f \mapsto \int_0^1 f(x)\,dx$$

Ex:

$$f(x) = \begin{cases} e^x, & x \geq 0 \\ 1-x, & x < 0 \end{cases}$$

Ex: $D \subseteq X$ metric space

$$C(D,\mathbb{R}) = \{ f : D \to \mathbb{R} \mid f \text{ cont.} \}$$
$$\|f\|_{\sup} = \sup_{x \in D} |f(x)|$$

normed vector space $\Downarrow$ metric space

D compact $\Rightarrow$ $C(D,\mathbb{R})$ is complete.

(any Cauchy sequence is convergent)

3 Derivatives

derivative

$f: \mathbb{R} \to \mathbb{R}$ $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\varepsilon_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i'th pos.

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad \frac{\partial f}{\partial x_i} = f_i' = \cancel{\lim \frac{f(x+\varepsilon_i h), x}{}}$

$$= \lim_{h \to 0} \frac{f(\underline{x} + \underline{\varepsilon}_i \cdot h) - f(\underline{x})}{h}$$

partial derivatives

$$\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}$$

Total derivative:

$f: \mathbb{R}^n \to \mathbb{R}$

For any $\underline{x} \in \mathbb{R}^n$, the total derivative $Df(\underline{x}) = A$ is a $1 \times n$-matrix s.t.:

For any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|y - x\| < \delta \Rightarrow |f(y) - f(x) - A \cdot (y - x)| < \varepsilon \cdot |y - x|$$

Alternatively:

$$\lim_{y \to x} \frac{\|f(y) - f(x) - A(y-x)\|}{\|y - x\|} = 0 \qquad A = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \right)$$

Defn: f is differentiable at $\underline{x}$ if $Df(\underline{x}) = A$ exists

i) f diff. at $\underline{x} \Rightarrow Df = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$

all partial derivatives exist

ii) If all partial derivatives $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ exists and are continuous around $\underline{x}$, then f is differentiable and $Df(\underline{x}) = (\partial f/\partial x_1, \ldots)$.

Defn: C^1-functions: partial derivatives exist and are continuous.

C^2-functions: all second order partial derivatives exist and are continuous.

$$\boxed{C^2 \Rightarrow C^1 \Rightarrow C = \text{continuous.}}$$

Thm: If $f : D \to \mathbb{R}$ is C^2, then ($D \subseteq \mathbb{R}^n$)

$$H(f) = \begin{pmatrix} f''_{11} & f''_{12} & \cdots & f''_{1n} \\ f''_{21} & f''_{22} & \cdots & f''_{2n} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

is symmetric.