Plan:

1. Overview of the course
2. Review of matrices and linear systems
3. Eigenvalues, eigenvectors and diagonalization
4. Quadratic forms and definiteness

1. Overview: 10 lectures
problem set for each lecture

See web page

(2) Matrices and linear systems

Matrices:

An $m \times n$ - matrix A: (m rows, n col's)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

A n-vector $\underline{v}$: (col. vector)

$$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Operations on matrices:

* Addition/subtraction: (A, B has same size)

$$A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$
$$A - B = (a_{ij} - b_{ij})$$

* Scalar multiplication:

$$c \cdot A = c \cdot (a_{ij}) = (c\, a_{ij})$$

c scalar (number)

* Multiplication: (#cols in A = #rows in B)

$$\underset{m \times n}{A} \cdot \underset{n \times p}{B} = (a_{ij}) \cdot (b_{ij}) = \underset{m \times p}{(c_{ij})}$$

where $c_{ij} = a_{i1} b_{1j} + \dots + a_{in} b_{nj}$

Note: $AB \neq BA$

* Transpose:

$$\underset{m \times n}{A} \rightsquigarrow \underset{n \times m}{A^T} \quad (= A^t, A^*)$$

$$A = (a_{ij}), \quad A^T = (a_{ji})$$

Note: $(AB)^T = B^T \cdot A^T$

Special matrices:

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad A+O = O+A = A$$

zero matrix

$$I = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \qquad A \cdot I = I \cdot A = A$$

(multiplicative unit)

identity matrix

Square matrix: $n \times n$, i.e. #rows = #cols

A $n \times n$-matrix $\quad A^2, A^3, \ldots, A^n, \ldots$

Symmetric matrix: $A^T = A$

Diagonal matrix: $D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$

Upper/lower triangular matrix: $A = \begin{pmatrix} d_1 & * & * & \cdots & * \\ 0 & d_2 & * & \cdots & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$ upper (similar for lower)

Note: ~~If~~ A, B are upper/lower triangular (in particular diagonal), then AB has the same type, and

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = (c_{ij}) \quad \text{with } c_{ii} = a_{ii} \cdot b_{ii}$$

Inverses: An inverse matrix of A is a matrix A^{-1} s.t. $A \cdot A^{-1} = I = A^{-1} \cdot A$.

If the inverse exists, it is unique.

Note: $(AB)^{-1} = B^{-1} \cdot A^{-1}$ if A, B invertible

$$AB \cdot B^{-1}A^{-1} = AA^{-1} = I$$

Determinants: $\underset{n \times n}{A} \rightsquigarrow \det(A) = |A|$ determinant of A (a number)

$A = (a)$ $|A| = a$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $|A| = ad - bc$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\Downarrow$$

$$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \cdots + a_{1n} C_{1n},$$

cofactor expansion

where $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

and M_{ij} = determinant of submatr. obtained by deleting row i, col. j.

cofactor

minor

Note: $|AB| = |A| \cdot |B|$

$|A^T| = |A|$

Important fact: $|A| \neq 0 \iff A^{-1}$ exists

If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{|A|} \cdot (C_{ij})^T$$

$n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad |A| = ad - bc$

$ad - bc \neq 0$: $\quad A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Linear systems:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\}$$ $m \times n$ linear system

$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ $\quad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

coeff. matrix

$A\underline{x} = \underline{b}$ $\quad$ linear system in matrix form

If $|A| \neq 0$, $\underline{x} = A^{-1} \cdot \underline{b}$ (one solution)

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} a_{11} & a_{12} \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} \dots & a_{mn} & b_m \end{array}\right)$$

augmented matrix

Gaussian elimination:

General method for solving linear systems using augmented matrix.

Ex:
$$x+y+z=3$$
$$x+2y+4z=7$$
$$x+3y+9z=13$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right)$$

-1 -1

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right)$$

-2

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form

pivot = first non-zero entry in a row

echelon form:

1. all zero rows are below other rows
2. all entries under a pivot must be zero

elementary row operations:

1. switch two rows
2. multiply a row with $c \neq 0$
3. add a multiple of one row to another row

$$x+y+z=3$$
$$y+3z=4$$
$$2z=2$$

backwards substitution

solve for basic variables, starting from the last equation.

$$x+2=3 \Rightarrow x=1$$
$$y+3=4 \Rightarrow y=1$$
$$z=1$$

One solution: $(x,y,z)=(1,1,1)$

Any matrix can be transformed into an echelon form using elementary row operations.

The echelon form is not unique, but the pivot positions (= positions of pivots in echelon form) are.

Reduced echelon form: echelon form s.t.

i) all pivots are 1

ii) all entries over a pivot are zero.

There is a unique reduced echelon form for any matrix.

Thm:

Any linear system has
i) one solution, or
ii) infinitely many solutions, or
iii) no solutions

Moreover:

no solutions $\Leftrightarrow$ pivot position in the last col.

otherwise: free variables $\leftrightarrow$ otherwise
basic variables $\leftrightarrow$ pivot pos. in corr. col.

one solution: no degrees of freedom = no free variables

infinitely many solutions: ≥ 1 degrees of freedom

Rank: Rank of A = # pivot positions in A

$\text{Nul}(A) = \{ \underline{x} : A\underline{x} = \underline{0} \}$ $\qquad A \cdot \underline{x} = \underline{0}$

nullspace of A $\qquad$ homogeneous linear system

Fact: $\boxed{\dim \text{Nul}(A) = n - \text{rk}(A)}$ when A is an $m \times n$-matrix.

Ex: $A = \begin{pmatrix} 1 & 2 & 7 & -1 \\ 0 & 0 & 3 & 4 \end{pmatrix}$

$x_1 + 2x_2 + 7x_3 - x_4 = 0$

$3x_3 + 4x_4 = 0$

$\left. \begin{array}{l} n = 4 \\ \text{rk} A = 2 \end{array} \right\}$ $4 - 2 = 2$ free variables

$x_1 + 7 \cdot \left(-\frac{4}{3} x_4\right) = -2x_2 + x_4$

$x_3 = -\frac{4}{3} x_4$

Nul(A) is two-dim., parametrized by x_2, x_4 $\longrightarrow$

$$\boxed{\begin{array}{l} x_1 = -2x_2 + \frac{31}{3} x_4 \\ x_3 = \qquad\quad\; -\frac{4}{3} x_4 \end{array}}$$

$\mathcal{B} = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\} \rightsquigarrow A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n)$

m-vectors mxn-matrix

Defn: $\mathcal{B}$ is linearly independent if

$\boxed{x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_n\underline{v}_n = \underline{0}}$ $\iff$ $A \cdot \underline{x} = \underline{0}$ homog. lin. syst.

has only the zero solution

$\underline{x} = \underline{0}$ $(x_1 = x_2 = \cdots = x_n = 0)$ $\iff$ $\operatorname{rk} A = n$

otherwise, $\mathcal{B}$ is linearly dependent.

Fact:

$\operatorname{rk} A = n$

$\Updownarrow$

$\mathcal{B}$ linearly independent

$\operatorname{rk} A$ = max. number of linearly independent vectors among the cols $\mathcal{B}$.

Note: $\boxed{\operatorname{rk} A = \text{maximal order of a non-zero minor in } A}$

minor = determinant of a submatrix.

$\underset{n\times n}{A}$ $\operatorname{rk} A = n \iff \det(A) \neq 0$

③ Eigenvalues and eigenvectors

A $n \times n$

Defn: $\underline{v}$ is a eigenvector for A, λ is an eigenvalue $\Leftrightarrow$ $A \cdot \underline{v} = \lambda \underline{v}$ (with $\underline{v} \neq \underline{0}$)

$$E_\lambda = \{ \underline{v} : A\underline{v} = \lambda \underline{v} \}$$

eigenspace of A with eigenvalue λ.

Computation:

$$A\underline{v} = \lambda \underline{v}$$
$$A\underline{v} - \lambda \underline{v} = \underline{0}$$
$$(A - \lambda I)\underline{v} = 0 \qquad \text{non-zero sol. for } \underline{v}$$

$\Updownarrow$

Char. eqn. for A polynomial of deg. n $\longrightarrow$ $\underline{\det(A - \lambda I) = 0}$

Solutions = eigenvalues

Facts:

A
$n \times n$
matrix

i) Eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_r$ with $r \leq n$

(each root is repeated according to its multiplicity)

ii) Eigenvectors:

$\dim E_\lambda \leq$ mult. λ

and

$1 \leq \dim E_\lambda \ldots$

λ^* has multiplicity m:

$\det(A - \lambda I) = 0$

$(\lambda - \lambda^*)^m$ is the highest power of $(\lambda - \lambda^*)$ in $\det(A - \lambda I)$.

Diagonalization: A is diagonalizable if there is an invertible matrix P and a diagonal matrix D s.t.

$$P^{-1} A P = D \iff A = PDP^{-1}$$

Appl: If A is diagonal, then

$$A^n = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}}) \cdots (\cancel{P}DP^{-1})$$

$$\boxed{A^n = P \cdot D^n \cdot P^{-1}}$$

Fact: A diagonalizable

$\Updownarrow$

i) n eigenvalues (counted with mult.)

ii) $\dim E_\lambda = m$ for any eigenvalue λ of mult. m.

In this case:

$$P = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n}) \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$(A\underline{v_i} = \lambda_i \underline{v_i})$ $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ lin. independent

Fact: i) A symmetric $\Longrightarrow$ A diagonalizable

ii) If A is diagonalizable:

$$\lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n = \det(A)$$
$$\lambda_1 + \lambda_2 + \dots + \lambda_n = \operatorname{tr}(A)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$\operatorname{tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$

4) Quadratic forms and definiteness

$$Q(x_1 \ldots x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + \cdots + c_{nn}x_n^2$$

quadratic form (where c_{ij} are numbers)

$$= \underline{x}^T \cdot A \cdot \underline{x} \quad , \quad A \text{ symm. } n \times n\text{-matrix}$$

Ex: $x_1^2 + 2x_1x_2 + 4x_1x_3 - x_2^2$ $\qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$= (x_1 \; x_2 \; x_3) \cdot \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$Q(\underline{x}) \longleftrightarrow A \text{ symm. matrix}$$

Defn:

$Q(\underline{x}) \geq 0$ for all $\underline{x}$: Q pos. semidefinite

$Q(\underline{x}) \leq 0$ — " — : Q neg. — " —

neither : Q indefinite

$Q(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$: Q pos. definite

$Q(\underline{x}) < 0$ — " — : Q neg. definite

pos. defn. global min

neg. defn. global max.

pos. semi-defn.

indefinite saddle pt.

Result 1:

A pos. semidefn $\Leftrightarrow$ $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$
pos. defn. $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$

A neg. semidefn $\Leftrightarrow$ $\lambda_1, \lambda_2, \ldots, \lambda_n \leq 0$
neg. defn. $\lambda_1, \ldots, \lambda_n < 0$

A indefinite $\Leftrightarrow$ both pos. and neg. eigenvalues

Result 2:

A pos. defn. $\Leftrightarrow$ $D_1, D_2, \ldots, D_n > 0$
A neg. defn. $\Leftrightarrow$ $D_1 < 0, D_2 > 0, \ldots, (-1)^n D_n > 0$

A pos. semidefinite $\Leftrightarrow$ $\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0$ for all principal minors

A neg. semidefn $\Leftrightarrow$ $\Delta_1 \leq 0, \Delta_2 \geq 0, \ldots, (-1)^n \Delta_n \geq 0$ for all principal minors

D_i: leading principal minors of A

Δ_i = principal minor of order i

$= M_{l_1 l_2 \ldots l_i, l_1 l_2 \ldots l_i}$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

$D_1 = M_{1,1} = 1$
$D_2 = M_{12,12} = -5$
$D_3 = M_{123,123} = |A| = -10$

RRC = reduced rank criterion (2017)

If A is a symmetric $n \times n$-matrix of $\text{rk}\, A = r < n$, then

$D_1, D_2, \ldots, D_r > 0 \Rightarrow A$ pos. semidefn.

$D_1 < 0, D_2 > 0, \ldots, (-1)^r D_r > 0 \Rightarrow A$ neg. semidefn.

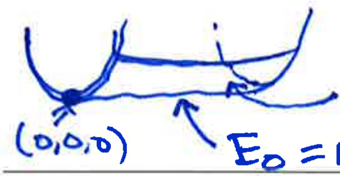
rk $A = 2$

Ex: $A = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 3 \end{pmatrix}$ $\quad 3x_1^2 + x_2^2 + 3x_3^2 - 6x_1x_3$

$$\left.\begin{aligned} D_1 &= 3 \\ D_2 &= 3 \\ D_3 &= 0 \end{aligned}\right\} \xRightarrow[\text{rk}\, A = 2]{\text{RRC}} A \text{ pos. semidefn.}$$

$$\left.\begin{aligned} \Delta_1 &= 3, 1, 3 \\ \Delta_2 &= 3, 3, 0 \\ \Delta_3 &= 0 \end{aligned}\right\} \Rightarrow A \text{ pos. semidefn.}$$

A pos. semidefn: $Q(x_1, x_2, x_3) = 3x_1^2 + x_2^2 + 3x_3^2 - 6x_1x_3 \geq 0$ for all (x_1, x_2, x_3)



graph of f $\Rightarrow (0,0,0)$ is a global min. pt of f (there are inf. many global min. pts.)