Solution: Mock exam 09/14 DKE 7017

1) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 14 \end{pmatrix}$

a) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} -5 & 1 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -14 \end{pmatrix}$

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}}$$

b) $\underline{z}' = \begin{pmatrix} -5 & 1 \\ 2 & -4 \end{pmatrix} \underline{z}$ with $\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x-1 \\ y-4 \end{pmatrix}$

$\underline{z} = c_1 \cdot \underline{v_1} \cdot e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t}$

$\begin{vmatrix} -5-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = \lambda^2 + 9\lambda + 18 = 0$

$\lambda_1 = -3 \quad \lambda_2 = -6$

$\underline{\lambda_1 = -3}$: $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \underline{v} = \underline{0}$ $\quad \underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\underline{\lambda_2 = -6}$: $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \underline{v} = \underline{0}$ $\quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{x} = \underline{z} + \begin{pmatrix} 1 \\ 4 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-6t} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$x = c_1 e^{-3t} - c_2 e^{-6t} + 1$

$y = 2c_1 e^{-3t} + c_2 e^{-6t} + 4$ //

c) Since $\lambda_1, \lambda_2 < 0$ the system is stable (glob. asympt. stable), since $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ when $t \to \infty$. At $t = 200$,

$$x = C_1 e^{-600} - C_2 e^{-1200} + 1 \simeq 1 = \bar{x}$$

$$y = 2C_1 e^{-600} + C_2 e^{-1200} + 4 \simeq 4 = \bar{y}$$

2. $f(x,y,z,w) = xw - yz$

a) $$H(f)(x,y,z,w) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$D_1 = 0$ $\quad \Delta_1 = 0, 0, 0, 0$

$D_2 = 0$ $\quad \Delta_2 = 0, 0, -1, \ldots$ $\quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$D_3 = 0$ $\quad \vdots$

$D_4 = 1$

rows = 1, 4
cols = 1, 4

$H(f)$ indefinite

f not convex
not concave

b) If there is a global max/min on $\mathbb{R}^4$, it must be a stationary pt. that is a local max/min. But this is not possible since $H(f)(\underline{x})$ is indefinite at all pts.

3. min $\underset{\displaystyle x^2+y^2+z^2}{f(x,y,z)}$ subj. to. $2x^2+6y^2+3z^2 \geq 36$

a) $D = \{(x,y,z) : 2x^2+6y^2+3z^2 \geq 36\}$ is not bounded.

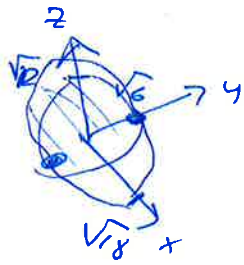
For example, $(x, \sqrt{6}, 0) \in D$ for all x since $2x^2+36 \geq 36$

b) $H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ pos. defn. since $\begin{cases} D_1 = 2 \\ D_2 = 4 \\ D_3 = 8 \end{cases}$

$\Rightarrow$ f convex, not concave

c)

Alt 1: D is the outside of an ellipsoid with semi-axes $(\sqrt{18}, \sqrt{6}, \sqrt{12})$



min $x^2+y^2+z^2$

$\Updownarrow$

min $\sqrt{x^2+y^2+z^2}$

$\Downarrow$

min $= 6$ at $(0, \pm\sqrt{6}, 0)$

Since this is the pt. on ellipsoid closest to origin.

Alt 2: Kuhn-Tucker formulation

max $-x^2-y^2-z^2$ s.t. $-2x^2-6y^2-3z^2 \leq -36$

$$\mathcal{L} = -x^2-y^2-z^2 - \lambda(-2x^2-6y^2-3z^2)$$
$$= -x^2-y^2-z^2+2\lambda x^2+6\lambda y^2+3\lambda z^2$$

FOC:
$$\begin{aligned} \mathcal{L}'_x &= -2x+4\lambda x = 0 \\ \mathcal{L}'_y &= -2y+12\lambda y = 0 \\ \mathcal{L}'_z &= -2z+6\lambda z = 0 \end{aligned} \Leftrightarrow \begin{aligned} 2x(2\lambda-1) &= 0 \\ 2y(6\lambda-1) &= 0 \\ 2z(3\lambda-1) &= 0 \end{aligned} \Leftrightarrow \begin{cases} x=0 \text{ or } \lambda = 1/2 \\ y=0 \text{ or } \lambda = 1/6 \\ z=0 \text{ or } \lambda = 1/3 \end{cases}$$

C: $2x^2+6y^2+3z^2 \geq 36$

CSC: $\lambda \geq 0$ and $\lambda(2x^2+6y^2+3z^2-36) = 0$

We solve FOC + C + CSC:

i) $x=y=z=0$: no solution (contradicts C)

ii) $x=y=0, z\neq 0$: $\lambda = 1/3$ (FOC)

$2x^2+6y^2+3z^2=36$ (CSC) $\Rightarrow z = \pm\sqrt{12}$

$\Rightarrow (0,0,\pm\sqrt{12}; 1/3)$ $f=12$

$x=z=0, y\neq 0$: $\lambda = 1/6$, $y=\pm\sqrt{6}$ $\Rightarrow$ $(0,\pm\sqrt{6},0; 1/6)$ $f=6$

$y=z=0, x\neq 0$: $\lambda = 1/2$, $x = \pm\sqrt{18}$ $\Rightarrow$ $(\pm\sqrt{18},0,0; 1/2)$ $f=18$

Best candidate for min. so far: $(x,y,z)=(0,\pm\sqrt{6},0)$ $\lambda=\frac{1}{6}$ $f=6$

This is global max (for K-T problem, so really min) since

$$\mathcal{L}(x,y,z; 1/6) = -x^2-y^2-z^2-\frac{1}{6}(-2x^2-6y^2-3z^2)$$

$$= -x^2-y^2-z^2+\frac{1}{3}x^2+y^2+\frac{1}{2}z^2$$

$$= -\frac{2}{3}x^2-\frac{1}{2}z^2 \text{ is concave}$$

Concl: Global min at $(x,y,z)=(0,\pm\sqrt{6},0)$ with $f=6$

4. $\max \sum_{0}^{T} (3-u_t)x_t^2$ s.t. $\begin{cases} x_{t+1} = x_t u_t \\ x_0 = 1 \\ u_t \in U = [1,3] \end{cases}$

a) T=3:

$$J_3(x) = \max_u (3-u)x^2 = 2x^2 \quad \text{with } \underline{u_3 = 1}$$

$$J_2(x) = \max_u (3-u)x^2 + J_3(ux)$$

$$= \max_u (3-u)x^2 + 2u^2x^2$$

$$= \max_u (3-u+2u^2)\,x^2$$

$$= 18x^2 \quad \text{with } \underline{u_2 = 3}$$

$$J_1(x) = \max_u (3-u)x^2 + 18u^2x^2$$

$$= \max_u x^2(3-u+18u^2)$$

$$= 18\cdot 3^2\cdot x^2 = 18\cdot 9\,x^2 = 162x^2 \quad \text{with } \underline{u_1 = 3}$$

$$J_0(x) = \max_u (3-u)x^2 + 162u^2x^2$$

$$= 162\cdot 3^2\, x^2 = 1458x^2 \quad \text{with } \underline{u_0 = 3}$$

$$J_0(x_0) = \max \sum_0^T (3-u_t)x_t^2 = 1458\cdot 1 = \underline{\underline{1458}}$$

b) For general T:

$$J_T(x) = \max_u (3-u)x^2 = 2x^2 \qquad u_T = 1$$

$$J_{T-1}(x) = \max_u (3-u)x^2 + 2u^2x^2$$

$$= 2\cdot 3^2\cdot x^2 = 18x^2 \qquad u_{T-1} = 3$$

⋮

Claim:

$$J_k(x) = 2 \cdot 9^{T-k} x^2 \quad , \quad k = T, T-1, \dots, 1, 0$$

Proof: ok for $k = T, T-1$ ✓

"induction" step:

Assume $J_{k+1}(x) = 2 \cdot 9^{T-(k+1)} x^2$ (assume ok for $k+1$)

Then:

$$J_k(x) = \max_u \; (3-u)x^2 + 2 \cdot 9^{T-(k+1)} u^2 x^2$$

$$= \max_u \left(3 - u + 2 \cdot 9^{T-(k+1)} u^2\right) x^2$$

$$= 2 \cdot 9^{T-(k+1)} \cdot 9 x^2 = 2 \cdot 9^{T-k} x^2 \quad \text{with } u_k = 3$$

(hence ok for k)

By "induction",

$$J_0(x) = 2 \cdot 9^T x^2 \Rightarrow J_0(x_0) = \underline{\underline{2 \cdot 9^T}}$$

is the maximum value