(2) Infinite Horizon dynamic programming ← This will be explained in lecture 10.

$$\max \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \quad \text{subject to} \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(x_t, u_t) \\ u_t \in \mathcal{U} \subseteq \mathbb{R} \end{cases}$$

Note:
* instead of $t \leq T$ we have infinite horizon $t \to \infty$
* $\beta^t f(x_t, u_t) = f(t, x_t, u_t)$; $g(x_t, u_t) = g(t, x_t, u_t)$
  ($0 < \beta < 1$ is the one-periode discount factor)
* we assume that $f(x_t, u_t)$ is bounded i.e. $|f(x_t, u_t)| < M$ for all t for some number $M > 0$. This implies that the sum

$$\sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \leq \sum_{t=0}^{\infty} \beta^t M = \frac{M}{1-\beta} \quad \text{is finite}$$

Bellman equation: Let $J(x) = J_0(x) = \max_{(u_t)} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t)$ sub. to ... $\begin{cases} x_0 = x \\ \vdots \end{cases}$

$$\boxed{J(x) = \max_{u \in \mathcal{U}} \left\{ f(x,u) + \beta J(g(x,u)) \right\}}$$

- Functional equations: we want to solve for the function $J(x)$. Difficult to find max when $J(x)$ is not known, hence difficult to find $J(x)$ by solving the Bellman equation.

- When $0 < \beta < 1$ and $|f(x_t, u_t)| < M$, the Bellman equation has a unique bounded solution $J^*(x)$. If we "guess" $\mathcal{J}(x)$ and it fits in the equation, it is therefore the unique solution.

Ex:
Problem 12.3.1 in [FMEA]. See also (SM) Student Manual (online)

Lecture 10

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DRE 7017
MATHEMATICS

PLAN:

1. Fixed points
2. Fixed point theorems.
3. Correspondences and Kakutanis thm.

Reading:

[FMEA] 14
[S] 9, 12

① Fixed points:

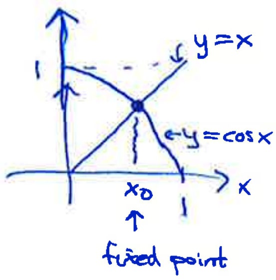
Let $f: X \to X$ be a function, where X is a set. A fixed point for f is an element $x \in X$ s.t. $f(x) = x$. A function $f: X \to X$ where the domain and the codomain is the same, is called an operator.

Ex: $f: [0,1] \to [0,1]$, $x \mapsto x^2$

Fixed points: $f(x) = \cancel{x}$
$x^2 = x$
$x = 0$, $x = 1$

$f: [0,1] \to [0,1]$, $x \longmapsto \cos x$

Fixed points: $f(x) = x$
$\cos x = x$
Solution?



Ex: A $n \times n$-matrix
A is an operator on $\mathbb{R}^n$ $\Big\}$ $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\underline{x} \longmapsto A \cdot \underline{x}$

Fixed point: A vector $\underline{x} \in \mathbb{R}^n$ s.t. $A\underline{x} = \underline{x}$

$\lambda = 1$ not eigenvalue: $\underline{x} = \underline{0}$ only fixed point

$\lambda = 1$ eigenvalue: $E_1 = \{\underline{x} : A\underline{x} = 1 \cdot \underline{x}\}$ are fixed points

Equilibrium states are fixed points for an appropriately constructed operator, in many cases.

$$x_{t+1} = \underline{A} x_t \longrightarrow \text{fixed points of } A: \mathbb{R}^n \to \mathbb{R}^n \text{ are equilibrium points}$$

(Markov chain)

(2) Sufficient conditions for existence of fixed points

Theorem (Brouwer):
If $K \subseteq \mathbb{R}^n$ is non-empty, convex and compact and $f: K \to K$ is a continuous operator on K, then f has a fixed point.

* Nonconstructive: Neither the theorem nor its proof says how to find the fixed point. There may be more than one.

Let (X, d) be a metric space (with metric d). A contraction mapping $f: X \to X$ is an operator such that

$$d(f(x), f(y)) \leq \beta \cdot d(x,y)$$

for all $x, y \in X$, where $\beta \in (0,1)$ is a constant (independent of x, y).

Ex: $\underset{[0,1]}{X} \xrightarrow{f} \underset{[0,1]}{X}$ is given by $f(x) = \frac{1}{2}x$, and $X = [0,1] \subseteq \mathbb{R}$ has the Euclidean metric. Then

$$d(f(x), f(y)) = d\left(\tfrac{1}{2}x, \tfrac{1}{2}y\right) = \left|\tfrac{1}{2}x - \tfrac{1}{2}y\right|$$

$$= \frac{1}{2}|x-y| = \frac{1}{2}d(x,y)$$

This is a contraction with $\beta = 1/2$.

Note: A contraction is continuous.

Theorem

Let $f : X \to X$ be a contraction on a complete metric space (X, d). Then f has a unique fixed point.

Proof:

If $x, y \in X$ are both fixed points, then $f(x) = x$ and $f(y) = y$. But if $x \neq y$,

$$\underbrace{d(f(x), f(y)) = d(x,y)}_{\text{since } x,y \text{ are fixed points}} \leq \beta\, d(x,y) \underbrace{<}_{\text{since } \beta<1 \text{ and } d(x,y)\neq 0} d(x,y)$$

This is a contradiction. So there is at most one fixed point.

If $x_0 \in X$ is any point, define $x_1 = f(x_0)$, $x_2 = f(x_1)$, $\ldots$, $x_{n+1} = f(x_n)$. Then $\{x_n\}$ is a sequence in X. We can prove that it is a Cauchy sequence:

$$d(x_2, x_1) = d(f(x_1), f(x_0)) \leq \beta\, d(x_1, x_0)$$

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \beta\, d(x_n, x_{n-1}) \quad \text{for } n \geq 1$$

$$\begin{aligned} d(x_{n+k}, x_n) &\leq d(x_{n+k}, x_{n+k-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \beta^{n+k-1} d(x_1, x_0) + \beta^{n+k-2} d(x_1, x_0) + \cdots + \beta^n d(x_1, x_0) \\ &= \beta^n \cdot \frac{1-\beta^k}{1-\beta} \cdot d(x_1, x_0) \\ &\leq \frac{\beta^n}{1-\beta} d(x_1, x_0) \end{aligned}$$

For n sufficiently large, $d(x_{n+k}, x_n) < \varepsilon$ for all k when $\varepsilon > 0$ is given.

Let $x = \lim x_n \in X$ (since X is complete). Then

$$d(f(x), x) = \lim_{n\to\infty} d(f(x_n), x_n) \leq \beta \lim_{n\to\infty} d(x_n, x_{n-1}) = \beta \lim_{n\to\infty} d(f(x_{n-1}), x_{n-1})$$

$$= \beta\, d(f(x), x) \quad \cancel{\text{and}}$$

This implies that $d(f(x), x) = 0$ since $\beta < 1$. Hence $f(x) = x$.

$\square$

Note: the proof of the contraction fixed point theorem is constructive.

For any $x_0 \in X$, the fixed point $x \in X$ is given by

$$x = \lim x_n \quad , \quad \text{where } x_n = f^n(x_0)$$

The fixed point is also unique.

Let $S \subseteq \mathbb{R}^n$.

* If $S \subseteq \mathbb{R}^n$ is compact, then $C(S,\mathbb{R})$ of cont. functions $f: S \to \mathbb{R}$ is a complete metric space with the sup norm. (Lecture 2)
* For any $S \subseteq \mathbb{R}^n$, $B(S,\mathbb{R}) = \{ f: S \to \mathbb{R} \text{ continuous and bounded} \}$ is a complete metric space with the sup norm.

$f: S \to \mathbb{R}$ bounded $\iff$ there is $M > 0$ s.t. $|f(s)| < M$ for all $s \in S$.
(or $-M < f(s) < M$)

Application: Bellman equation

$$J(x) = \max_{u \in U} \{ f(x,u) + \beta J(g(x,u)) \}$$

Conditions:
$\beta < 1$
f bounded.
f, g cont.

Consider $X = B(\mathbb{R},\mathbb{R})$, and the operator

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & X \\ J & \longmapsto & \max\limits_{u \in U} \{ f(x,u) + \beta J(g(x,u)) \} \end{array}$$

One may show that T is a well-defined operator (if J is bounded, then $T(J)$ is well-defined bounded function) and that T is a contraction.

Hence T has a fixed point $J(x)$ that is unique.
satisfies Bellman equation.

(Note: $T(J) = J \iff J(x)$

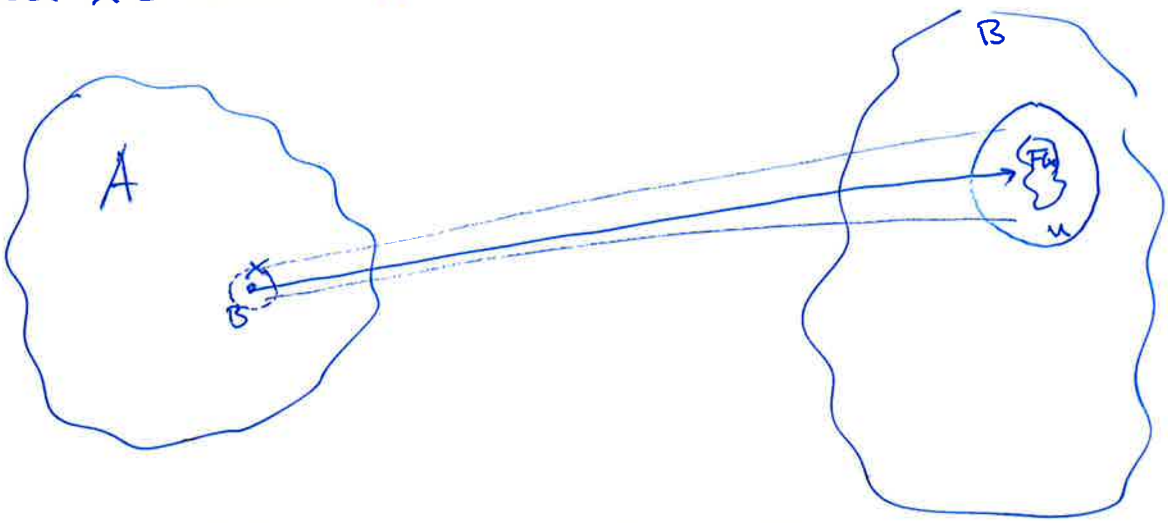
Correspondences

A correspondence $F: A \twoheadrightarrow B$ is a specification of a subset $F(x) \subseteq B$ for all $x \in A$. That is, a multivalued function.
The graph of F is

$$\Gamma(F) = \{(x,y) \in A \times B : x \in A,\ y \in F(x)\}$$

F is upper hemicontinuous if the following condition is satisfied:

For any $x \in A$, and for any open set U containing $F(x)$, there is an open ball $B(x,r)$ around x such that $F(x') \subseteq U$ for all $x' \in B(x,r) \cap A$.



If the graph of F is compact, then F is upper hemicont.

Theorem (Kakutani)
If $K \subseteq \mathbb{R}^n$ is nonempty, compact and convex and if $F: K \to K$ is upper hemicontinuous such that $F(x)$ is nonempty and convex for all $x \in K$, then F has a fixed point $x \in K$ (i.e. $F(x) \ni x$).