Eivind Eriksen

Notes on Euclidean Spaces
DRE 7017 Mathematics, PhD

August 8, 2014

BI Norwegian Business School

Contents

Chapter 1
Basic Notions

1.1 Sets

A *set* S is a well-specified collection of elements. We write $x \in S$ when x is an element of S, and $x \notin S$ otherwise. The *empty set* is the set with no elements, and it is denoted $\emptyset$. We say that T is a subset of S, and write $T \subseteq S$, if any element of T is also an element of S.

We may specify a set by listing its elements, either completely (for finite sets) or by indicating a pattern (for countable sets). Typical examples of sets specified by a list of elements are $S = \{1,2,3,4,5,6\}$, $T = \{1,2,3,\dots,100\}$ and $\mathbb{N} = \{1,2,3,\dots\}$. Another usual way to specify a set is to use a property, such as

$$S = \{x : x \text{ is divisible by } 3\}, \quad T = \{(x,y) : x^2 + y^2 \leq 1\}$$

The set S is the set of all numbers x such that x is divisible by 3, and the set T is the set of points (x,y) such that $x^2 + y^2 \leq 1$. We use the following standard notation for some important sets:

1. $\mathbb{N} = \{1,2,3,\dots\}$ is the set of *natural numbers*.
2. $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ is the set of *integers*.
3. $\mathbb{Q} = \{\frac{p}{q} : p,q \in \mathbb{Z}, q \neq 0\}$ is the set of *rational numbers*.
4. $\mathbb{R} = \{x : x \text{ is a real number }\}$ is the set of *real numbers*.

The rational numbers have a decimal representation that is either finite, or eventually periodic. Numbers with decimal representations that are not eventually periodic are called *irrational*, and the real numbers are the number that are either rational or irrational. (For a more precise definition, see Appendix B in [2]).

Given sets S,T, we define their *union* $S \cup T$, their *intersection* $S \cap T$ and their *difference* $S \setminus T$ in the following way:

1. $S \cup T = \{x : x \in S \text{ or } x \in T\}$
2. $S \cap T = \{x : x \in S \text{ and } x \in T\}$
3. $S \setminus T = \{x : x \in S \text{ and } x \notin T\}$

When S is a subset of a given set U, then the difference $U \setminus S$ is called the *complement* of S (in U), and is often written $S^c = U \setminus S$. The *Cartesian product* of the sets S, T is written $S \times T$, and is defined to be

$$S \times T = \{(s,t) : s \in S,\ t \in T\}$$

We often use the notation $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) : x,y \in \mathbb{R}\}$, and more genererally $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1,x_2,\ldots,x_n) : x_1,x_2,\ldots,x_n \in \mathbb{R}\}$.

1.2 Logic

Let P and Q be statements, which could either be true or false. We say that P *implies* Q, and write $P \implies Q$, if the following condition holds: Whenever statement P is true, statement Q is also true. We write $P \Leftrightarrow Q$, and say that P and Q are *equivalent*, if $P \implies Q$ and $Q \implies P$.

Let us consider an implication $P \implies Q$. It is logically the same as the implication

$$\text{not } Q \implies \text{not } P$$

The second form of the implication is called the *contrapositive form*. For instance, a function f is called injective if the following condition holds:

$$f(x) = f(y) \implies x = y$$

This implication can be replaced with its contrapositive form, which is the implication

$$x \neq y \implies f(x) \neq f(y)$$

Mathematical arguments and proofs are sometimes easier to understand when implications are replaced with their contrapositive forms.

1.3 Numbers

Let $D \subseteq \mathbb{R}$ be a set of (real) numbers. We say that M is an *upper bound* for D if $x \leq M$ for all $x \in D$, and that s is a *least upper bound* or *supremum* for D if the following conditions hold:

1. s is an upper bound for D.
2. If s' is another upper bound for D, then $s' > s$.

We write $s = \sup D$ when s is a supremum for D. It is a very useful fact about the real numbers that any subset $D \subseteq \mathbb{R}$ with an upper bound has a supremum. If D does not have an upper bound, we write $\sup D = \infty$. Similar results holds for lower bounds, and the greatest lowest bound is called *infinum* and written $\inf D$. For example, when $D = \{1/n : n \in \mathbb{N}\} = \{1, 1/2, 1/3, \dots\} \subseteq \mathbb{R}$, then $\sup D = 1$ and $\inf D = 0$.

References

Appendix A.1-A.2 in Sydsæter et al [3]; Appendix A1 in Simon, Blume [1]; Appendix A in Sundaram [2].

Chapter 2
Euclidean Spaces

For any positive integer $n \geq 1$, the set $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$ is called the n-dimensional *Euclidean space*. An element $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a *point* or a *vector*.

2.1 Euclidean space as a vector space

For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any scalar (number) $r \in \mathbb{R}$, we define the *vector addition* $\mathbf{x} + \mathbf{y}$ and the *scalar multiplication* $r\,\mathbf{x}$ in $\mathbb{R}^n$ by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \quad r\,\mathbf{x} = (rx_1, rx_2, \ldots, rx_n)$$

The zero vector in Euclidean space is $\mathbf{0} = (0, 0, \ldots, 0)$, and the additive inverse of a vector $\mathbf{x} \in \mathbb{R}^n$ is the vector

$$-\mathbf{x} = (-x_1, -x_2, \ldots, -x_n)$$

The following conditions holds in Euclidean space: For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and for all numbers $r, s \in \mathbb{R}$, we have that

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
3. $\mathbf{x} + \mathbf{0} = \mathbf{x}$
4. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
5. $(rs)\mathbf{x} = r(s\mathbf{x})$
6. $r(\mathbf{x} + \mathbf{y}) = r\,\mathbf{x} + r\,\mathbf{y}$
7. $(r + s)\mathbf{x} = r\,\mathbf{x} + s\,\mathbf{x}$
8. $1 \cdot \mathbf{x} = \mathbf{x}$

The definition of a *vector space* is a set V (whose elements are called vectors), together with well-defined operations of vector addition and scalar multiplication in V, such that the conditions above hold. This means that in particular, Euclidean space $\mathbb{R}^n$ is a vector space.

There are also other vector spaces than Euclidean space $\mathbb{R}^n$. For instance, the set of all continuous functions defined on the interval $I = [0,1]$ is a vector space. If f, g are continuous functions defined on I and $r \in \mathbb{R}$ is a scalar, we define the functions $f+g$ and rf by

$$(f+g)(x) = f(x) + g(x), \quad (rf)(x) = rf(x)$$

for all $x \in I$. These operations are well-defined, and satisfy the conditions above. Therefore, the space $C(I, \mathbb{R})$ of continuous functions on the interval I is a vector space.

2.2 Inner products

The Euclidean *inner product* $\langle \mathbf{x}, \mathbf{y} \rangle$ of the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

It is also called the dot product or scalar product, and we often write $\mathbf{x} \cdot \mathbf{y}$ for $\langle \mathbf{x}, \mathbf{y} \rangle$. Notice that the result of the inner product is a scalar (a number). The Euclidean inner product satisfy the following conditions:

1. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
2. $(a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{z} = a\, \mathbf{x} \cdot \mathbf{z} + b\, \mathbf{y} \cdot \mathbf{z}$
3. $\mathbf{x} \cdot \mathbf{x} \geq 0$, and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$

In a general vector space, an inner product is a product which satisfy the above conditions.

Theorem 2.1 (Cauchy-Schwartz inequality). *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*, we have that*

$$|\mathbf{x} \cdot \mathbf{y}| \leq (\mathbf{x} \cdot \mathbf{x})^{1/2} (\mathbf{y} \cdot \mathbf{y})^{1/2}$$

2.3 Norms

The Euclidean *norm* of a vector $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$\|\mathbf{x}\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} = (\mathbf{x} \cdot \mathbf{x})^{1/2}$$

Notice that the norm of a vector is a (non-negative) scalar. If $n = 1$, then the norm $\|\mathbf{x}\| = |x|$, and if $n = 2$, then the norm $\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$ is given by Pytagoras' Theorem. The Euclidean norm satisfies the following conditions:

1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
2. $\|r\,\mathbf{x}\| = |r|\,\|\mathbf{x}\|$
3. $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

The last inequality is called the *triangle inequality*. In a general vector space, a norm is a function that satisfies the above conditions.

2.4 Metrics

The Euclidean *distance* between the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is given by

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

The distance function $d(\mathbf{x},\mathbf{y})$ is also called a *metric*. The Euclidean metric satsify the following conditions:

1. $d(\mathbf{x},\mathbf{y}) \geq 0$, and $d(\mathbf{x},\mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$
2. $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$
3. $d(\mathbf{x},\mathbf{z}) \leq d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z})$

The last inequality is called the triangle inequality. In general, a metric is a function that satisfy the conditions above, and a *metric space* (X,d) is a set X equipped with a metric d. In particular, Euclidean space is a metric space.

2.5 Sequences

Let X be a metric space with metric d. A *sequence* in X is a collection of points $x_i \in X$ indexed by the positive integers $i \in \mathbb{N} = \{1,2,3,\dots\}$. We usually write $x_1, x_2, x_3, \dots$ for such a sequence, or (x_i) in more compact notation.

In many applications, $X = \mathbb{R}^n$ is Euclidean space with the Euclidean metric d. But we will also consider sequences in other metric spaces.

Let (x_i) be a sequence in X. We say that (x_i) *converges* to a limit $x \in X$, and write $x_i \to x$ or $\lim x_i = x$, if the distance $d(x_i,x)$ between x and x_i tends to zero as i goes towards infinity. We may express this more precisely in the following definition:

Definition 2.1. The sequence (x_i) has limit x if the following condition holds: For every $\varepsilon > 0$, there is a positive integer N such that $d(x_i,x) < \varepsilon$ when $i > N$.

As an example, let us consider the sequence given by $x_i = 1/i$. This is a sequence in $\mathbb{R}$, the 1-dimensional Euclidean space, explicitly given by

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$$

Then (x_i) has limit $x = 0$, since $x_i = 1/i$ tends towards zero when i goes towards infinity. Indeed, if $\varepsilon > 0$ is given, we have that $d(x_i, x) = |1/i - 0| = 1/i < \varepsilon$ for $i > N$ when we choose $N > 1/\varepsilon$.

We say that the sequence (x_i) is *bounded* if there is a positive number $M \in \mathbb{R}$ and a point $p \in X$ such that $d(x_i, p) < M$ for all i. The set $\{x : d(x,p) < M\}$ is called the *open ball* around p with radius M, and is written $B(p,M)$.

Proposition 2.1. *Let (x_i) be a sequence in X. Then we have:*

1. *If (x_i) converges to x and to x', then $x = x'$.*
2. *If (x_i) converges, then it is bounded.*
3. *If (x_i) converges, then the* Cauchy criterion *holds: For any $\varepsilon > 0$, there is a positive integer N such that $d(x_k, x_l) < \varepsilon$ when $k, l > N$.*

It is easier to check the Cauchy criterion than to find the limit of a sequence. A sequence that satisfies the Cauchy criterion is called a *Cauchy sequence*.

Theorem 2.2. *Let $X = \mathbb{R}^n$ be Euclidean space, with the Euclidean metric. If a sequence (x_i) in X is a Cauchy sequence, then it converges to a limit $x \in X$.*

Notice that this theorem holds for Euclidean space $X = \mathbb{R}^n$, but not for all metric spaces. We say that a metric space if *complete* if every Cauchy sequence converges.

Let (x_i) be a sequence in X. A *subsequence* of (x_i) is a sequence obtained by picking an infinite number of elements from (x_i). More precisely, it is a sequence

$$x_{j_1}, x_{j_2}, x_{j_3}, \dots, x_{j_k}, \dots$$

defined by an infinite sequence of indices $j_1 < j_2 < j_3 < \dots < j_k < \dots$ in $\mathbb{N}$. If (x_i) converges to x, then any subsequence also converges to x. However, the opposite implication does not hold. For instance, the alternating sequence $1, -1, 1, -1, \dots$ has converging subsequences, but it is not convergent.

2.6 Topology

Let (X,d) be a topological space. Usually $X = \mathbb{R}^n$ is Euclidean space with the Euclidean metric, but we will also consider other metric spaces.

A subset $D \subseteq X$ is called *open* if the following condition holds: For any point $p \in D$, there is an open ball $B(p,M)$ around p that is contained in D. This means that if $p \in D$, then any point sufficiently close to p is also in D. Typical examples of open sets are open intervals $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ in $\mathbb{R}$, and more generally, open balls $B(p,M)$ in a metric space X.

Let $D \subseteq X$ be a subset. A point $p \in D$ is called a *boundary point* for D if any open ball $B(p,M)$ contains points in D and in D^c; that is, if $B(p,M) \cap D \neq \emptyset$ and $B(p,M) \cap D^c \neq \emptyset$ for all $M > 0$. The set of boundary points of D is written ∂D. A point $p \in D$ that is not a boundary point is called an *interior point*. We write $D^o = D \setminus \partial D$ for the interior points of D.

A subset $D \subseteq X$ is called *closed* if the complement $D^c = X \setminus D$ is open. Typical examples of closed sets are closed intervals $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ in $\mathbb{R}$, and more generally, closed balls $\overline{B}(p,M) = \{x \in X : d(x,p) \leq M\}$.

Proposition 2.2. *Let $D \subseteq X$ be a subset of a metric space X. Then we have:*

1. *D is open if and only if $\partial D \cap D = \emptyset$*
2. *D is closed if and only if $\partial D \subseteq D$*

A subset $D \subseteq X$ is *bounded* if there is a point $p \in D$ and a radius $M > 0$ such that $D \subseteq B(p,M)$, and it is *compact* if the following condition holds: Any sequence (x_i) in D has a subsequence that converges to a limit $x \in D$. It follows that a compact subset in X is closed and bounded.

Theorem 2.3 (Bolzano-Weierstrass). *Let (X,d) be the Euclidean space $X = \mathbb{R}^n$ with the Euclidean metric. Then a subset $D \subseteq X$ is compact if and only if D is closed and bounded.*

References

Appendix A.3 and Chapter 13.1 - 13.2 in Sydsæter et al [3]; Chapter 10, 12, 29 in Simon, Blume [1]; Chapter 1.1 - 1.2 and Appendix C in Sundaram [2].

References

1. Simon, C., Blume, L.: Mathematics for Economists. Norton (1994)
2. Sundaram, R.: A First Course in Optimization Theory. Cambridge University Press (1976)
3. Sydsæter, K., Hammond, P., Seierstad, A., Strøm, A.: Further Mathematics for Economic Analysis. Prentice Hall (2008)