Eivind Eriksen

Linear Systems and Gaussian Elimination

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BI Norwegian Business School

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Chapter 1
Linear Systems

1.1 Linear Equations

A *linear equation* in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where $a_1, a_2, a_3, \ldots, a_n$ and b are fixed numbers. These fixed numbers are called *parameters*. Typical linear equations are

$$x_1 + 2x_2 = 4 \quad \text{and} \quad 7x_1 + 3x_2 - x_3 = 0$$

The equations are called linear because their graphs are straight lines (when it is an equation in two variables) and planes (when it is an equation in three variables). Typical equations that are *not* linear are

$$x_1^2 - x_1x_2 = 1 \quad \text{and} \quad \ln x_1 - \sqrt{x_2} = 0$$

The key feature of a linear equations is that each term of the equation is either a constant term or a term of order one (that is, a constant coefficient times one of the variables).

There are several reasons to study linear equations. First, the linear equations are the simplest equations we have. The study of linear equations requires no calculus, and builds on techniques from school mathematics, such as the solution of two linear equations in two variables via substitution:

Example 1.1. We solve the following linear equations using substitution:

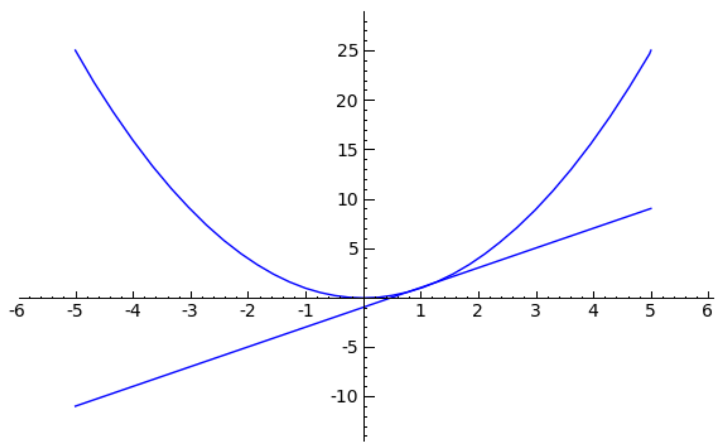
$$\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned}$$

First, we solve the first equation for y. Then we substitute $y = 4 - x$ into the second equation and solve for x.

$$\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned} \quad \Rightarrow \quad y = 4 - x \quad \Rightarrow \quad x - (4 - x) = 2 \quad \Rightarrow \quad x = 3$$

The solution is $x = 3$ and $y = 4 - x = 1$.

The real world is often non-linear and can in many cases best be described with more complicated, non-linear equations. But even in these cases, linear equations can be very useful: Using calculus, we may replace non-linear equations with *linear approximations*. A well-known and typical example is when we use the derivative of a function in one variable to approximate the graph of the function (a curve) with its tangent line at a given point. The tangent line is the best linear approximation of the function near that point.



Of course, the tangent line is not quite the same as the curve, especially far away from the point we used to construct the tangent. Nevertheless, linear approximations can very often be used successfully. Few would disagree that the earth is curved, and has the shape of a sphere (more or less). But it still makes sense to assume that it is flat when it comes to constructing buildings, and even cities.

1.2 Linear Systems

A *linear system* in the variables $x_1, x_2, \dots, x_n$ is a collection of one or more linear equations in these variables. The general form of a linear system with m equations in n variables is

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

where $a_{11}, a_{12}, \dots, a_{mn}$ and $b_1, b_2, \dots, b_m$ are fixed numbers called parameters. A linear system with m equations and n variables is called an $m \times n$ linear system.

Example 1.2. A company in the US earns before-tax profits of \$100,000. It has agreed to contribute 10% of its after-tax profits to a charity. It must pay state tax of 5% of its profit (after the donation) and a federal tax of 40% of its profit (after the donation and state taxes are paid). How much will the company pay in state and federal taxes, and how much will it donate?

Let S and F denote the state and federal taxes, and let C be the charity donation. The state tax is 5% of profits net of the donation, hence

$$S = 0.05(100,000 - C) \quad \Rightarrow \quad S + 0.05C = 5,000$$

The federal tax is 40% of profits net of the donation and state tax, so

$$F = 0.40(100,000 - C - S) \quad \Rightarrow \quad 0.40S + F + 0.40C = 40,000$$

Since the donation is 10% of after-tax profits, we must have

$$C = 0.10(100,000 - S - F) \quad \Rightarrow \quad 0.10S + 0.10F + C = 10,000$$

This gives the following linear system:

$$\begin{array}{rcrcrcr} S & & & + & 0.05C & = & 5,000 \\ 0.40S & + & F & + & 0.40C & = & 40,000 \\ 0.10S & + & 0.10F & + & C & = & 10,000 \end{array}$$

Note that when we wrote down this linear system, we chose an order of the variables, and then organized all the equations according to that order.

We remark that when the variables and their order is given, a linear system can be completely described by a rectangular table of numbers, called a *matrix*. The *coefficient matrix* of a linear system is the matrix consisting of all coefficients in front of the variables, and is often denoted A. The *augmented matrix* of the linear system is a matrix that contains the entries in the coefficient matrix, extended with the constants on the right side of each equation, and it is often denoted $\hat{A}$. The linear system

$$\begin{aligned} S \qquad\quad + 0.05C &= 5,000 \\ 0.40S + \quad F + 0.40C &= 40,000 \\ 0.10S + 0.10F + \quad C &= 10,000 \end{aligned}$$

of Example 1.2 has coefficient matrix A and augmented matrix $\hat{A}$ given by

$$A = \begin{pmatrix} 1 & 0 & 0.05 \\ 0.40 & 1 & 0.40 \\ 0.10 & 0.10 & 1 \end{pmatrix} \quad \text{and} \quad \hat{A} = \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0.40 & 1 & 0.40 & 40,000 \\ 0.10 & 0.10 & 1 & 10,000 \end{array}\right)$$

The augmented matrix is a compact notation that allows us to write down all the parameters of a linear system in a convenient way.

1.3 Solutions of Linear Systems

Let us consider the general $m \times n$ linear system in the variables $x_1, x_2, \ldots, x_n$, which is of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

A *solution* of this linear system is an n-tuple of numbers $(s_1, s_2, \ldots, s_n)$ such that $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ solves all m equations simultaneously.

For instance, the linear system in Example 1.1 has the solution $(x,y) = (3,1)$ since $x = 3$ and $y = 1$ solves both equations of the linear system

$$\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned}$$

In fact, both sides of the first equation evaluate to 4 and both sides of the second equation evaluate to 2 when we substitute $x = 3$ and $y = 1$. Another example is the linear system of Example 1.2, given by

$$\begin{aligned} S \qquad\quad + 0.05C &= 5,000 \\ 0.40S + \quad F + 0.40C &= 40,000 \\ 0.10S + 0.10F + \quad C &= 10,000 \end{aligned}$$

We have not solved this linear system yet. A number of questions come to mind when we are faced with an unsolved linear system such as this one:

- Does the linear system have solutions?

- If so, how many solutions are there?
- How do we find these solutions?

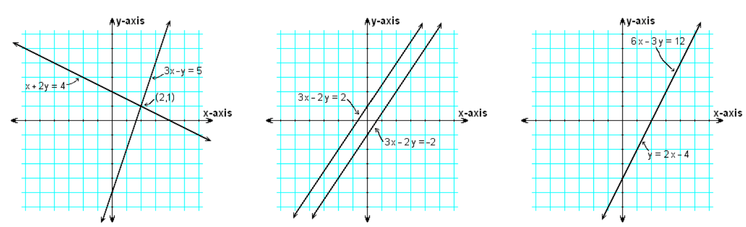
The first two questions hint at the fact that not all linear systems have a unique solution. Let us consider the general 2×2 linear system

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

to illustrate this. Each equation has a graph that is a straight line in the plane, and the intersection points of the two lines are the solutions of the linear system. There are three possible types of configurations:

- When the two lines are not parallel, there is a unique intersection point and a unique solution of the linear system.
- When the two lines are parallel and different, there are no intersection points and no solutions of the linear system.
- When the two lines are parallel and coincide, the intersection points are all the points on the line, and there are infinitely many solutions of the linear system.

In other words, if there are at least two solutions, then there must be infinitely many.



When you move from a 2×2 linear system to a general $m \times n$ linear system, the graphical picture is very different (if you can imagine pictures in n-dimensional space at all). However, the same threefold nature of solutions persists:

Theorem 1.1. *Any linear system has either no solutions, a unique solution or infinitely many solutions.*

Concerning the question of how to find the solutions of a linear system, there are a number of different approaches. The most commonly used methods can be characterized as *substitution methods*, *elimination methods*, and *matrix methods*.

When we use substitution to solve an $m \times n$ system, we first solve one of the equations for one of the variables — let us say we solve the first equation for x_n, so that x_n is expressed in terms of the other $n-1$ variables. Then we substitute this expression for x_n into the remaining $m-1$ equations. The result is a reduction of the original $m \times n$ linear system to a new $(m-1) \times (n-1)$ linear system — a system of $m-1$ equations in the $n-1$ variables $x_1, x_2, \ldots, x_{n-1}$. We continue with this kind of reduction until we have a system of a single equation that we can solve.

Example 1.3. We use substitution to solve the 3×3 linear system given by

$$\begin{aligned} S \quad\quad\quad\;\; + 0.05C &= \;\; 5,000 \\ 0.40S + \quad\; F + 0.40C &= 40,000 \\ 0.10S + 0.10F + \quad\; C &= 10,000 \end{aligned}$$

We first solve the first equation for S (since F is not present in this equation and the coefficient of S is 1, this gives easier calculations) and get

$$S = 5000 - 0.05C$$

Then we substitute the right hand side expression for S in the second and third equation. We get a 2×2 linear system

$$\begin{aligned} 0.01F + 0.995C &= \;\; 9,500 \\ F + \;\; 0.38C &= 38,000 \end{aligned}$$

The next step is to solve the second equation of the new system for F (again, this gives the easiest calculations since the term with F in the second equation has coefficient 1) and get

$$F = 38,000 - 0.38C$$

Then we substitute the right hand side expression for F in the first equation of the new linear system. We get

$$0.01(38,000 - 0.38C) + 0.995C = 9,500 \quad \Rightarrow \quad 0.957C = 5,700$$

This gives the solution $C = 5,956$. Then we substitute this value into the equation solved for F, and get $F = 35,737$. Finally, we substitute both these values into the equation solved for S, and get $S = 4,702$. The solution of the linear system is therefore $C = 5,956$, $S = 4,702$, $F = 35,737$ (rounded to the nearest dollar).

Although it is possible to use substitution to solve any linear system, it turns out that *elimination methods* are better suited for theoretical analysis than substitution methods. Elimination is the subject of the next chapter.

Problems

1.1. Net cost of charity donation

What would the state and federal tax be for the company in Example 1.2 if no charity donation was made? Use this to find the net cost for the company of the charity contribution of \$5,956.

1.2. From linear system to augmented matrix
Write down the coefficient matrix and the augmented matrix of the following linear systems:

$$a)\quad \begin{aligned} 2x + 5y &= 6 \\ 3x - 7y &= 4 \end{aligned} \qquad b)\quad \begin{aligned} x + y - z &= 0 \\ x - y + z &= 2 \\ x - 2y + 4z &= 3 \end{aligned}$$

1.3. From augmented matrix to linear system
Write down the linear system in the variables x, y, z with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 2 & -3 & 1 & 0 \\ 7 & 4 & 1 & 3 \end{array}\right)$$

1.4. Solution by substitution
Use substitution to solve the linear system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned}$$

1.5. Variation of parameters by substitution
For what values of h does the following linear system have solutions?

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + z &= h \end{aligned}$$

Chapter 2
Gaussian Elimination

2.1 Elimination

When we solve a linear system using *elimination*, we first replace the given linear system with a sequence of simpler linear systems by eliminating variables, making sure to only use *allowed operations* in each step (that is, operations that preserve the solutions of the linear system).

Let us start by a simple example of elimination, of the type that should be familiar from school mathematics:

Example 2.1. We use elimination to solve the 2×2 linear system given by

$$\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned}$$

We eliminate x by multiplying the last equation by -1 and then adding the two equations:

$$\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x + y &= 4 \\ -x + y &= -2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x + y &= 4 \\ 2y &= 2 \end{aligned}$$

Notice that we kept the first equation as part of the system. When the system is reduced to the simple form where x is eliminated from the second equation, we can find the solutions by a method called *back substitution*: We first solve the last equation for y;

$$2y = 6 \quad \Rightarrow \quad y = 3$$

Then we substitute $y = 3$ in the first equation and solve for x

$$x + 3 = 4 \quad \Rightarrow \quad x = 1$$

Notice that when we use elimination, we work with the entire linear system and not just one equation at the time. It is customary to use the augmented matrix to represent the linear system at each step, so that the notation is more compact and easier to work with:

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1 & 1 & 4 \\ -1 & 1 & -2 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 2 & 2 \end{array}\right)$$

2.2 Elementary row operations

The rows of the augmented matrix corresponds to the equations of the linear system, and to perform row operations on the matrix (that is, operations on the rows of the matrix) corresponds to performing operations on the equations of the system. The following row operations are called *elementary row operations*:

- to interchange two rows of the matrix
- to change a row by adding to it a multiple of another row
- to multiply each element in a row by the same non-zero number

We remark that these row operations preserve the solutions of the linear system. In fact, these row operations corresponds to switching two equations, to add a multiple of one equation to another equation, and to multiply an equation with a non-zero number. We say that two linear systems are *row equivalent* if you can get from one to the other using elementary row operations.

Our aim is to use elementary row operations to obtain a new and simpler form of the linear system — a linear system that is so simple that it can easily be solved by back substitution. Since elementary row operations preserve the solutions of linear systems, we can find the solutions of the given linear system using this process.

Notice that in Example 2.1, we used two elementary row operations to eliminate the variable x from the second equation. In fact, we could have combined the two operations into one elementary row operation: We replaced the second row with the sum of the second row and (-1) times the first row. With symbols, the elementary row operation was

$$R_2 \leftarrow R_2 + (-1)R_1$$

It was the presence of x in the first equation that allowed us to eliminate x from the second equation. Here is another example, with three variables:

Example 2.2. We use elementary row operations to simplify the 3×3 linear system given by

$$\begin{aligned} S &+ 0.05C = 5{,}000 \\ 0.40S + F &+ 0.40C = 40{,}000 \\ 0.10S + 0.10F &+ C = 10{,}000 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5{,}000 \\ 0.40 & 1 & 0.40 & 40{,}000 \\ 0.10 & 0.10 & 1 & 10{,}000 \end{array}\right)$$

In the first step, we use the presence of the variable C in the first equation to eliminate C from the second and third equation by means of two elementary row operations. In the second step, we use the presence of S in the second equation of the new system to eliminate S from the third equation:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0.40 & 1 & 0.40 & 40,000 \\ 0.10 & 0.10 & 1 & 10,000 \end{array}\right) \qquad \begin{array}{l} \\ R_2 \leftarrow R_2 + (-0.40)R_1 \\ R_3 \leftarrow R_3 + (-0.10)R_1 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0 & 1 & 0.38 & 38,000 \\ 0 & 0.10 & 0.995 & 9,500 \end{array}\right) \qquad \begin{array}{l} \\ \\ R_3 \leftarrow R_3 + (-0.10)R_2 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0 & 1 & 0.38 & 38,000 \\ 0 & 0 & 0.957 & 5,700 \end{array}\right)$$

The coefficients -0.40, -0.10 and -0.10 were chosen in a such a way that the correct variables were eliminated.

Finally, we use back substitution to solve the simplified system (that is, we use the the last augmented matrix): The last equation is

$$0.957C = 5,700 \quad \Rightarrow \quad C = 5,956$$

Then we substitute this value for C in the second equation, and get

$$F + 0.38(5,956) = 38,000 \quad \Rightarrow \quad F = 35,737$$

Finally, we substitute the values for F and C into the first equation, and find

$$S + 0.05(5,956) = 5,000 \quad \Rightarrow \quad S = 4,702$$

This gives the solution $S = 4,702$, $F = 35,737$, $C = 5,956$ (rounded to the nearest dollar).

2.3 Echelon and reduced echelon forms

How do we know when to stop the process of elementary row operations? In other words, what are the key features of an augmented matrix where as many variables as possible have been eliminated? To answer this question, we introduce some useful concepts for matrices.

The first non-zero entry in a row of a matrix is called a *pivot*. Hence each row will have one pivot, unless the row is a row of zeros. We say that a matrix has *echelon form* if all entries under a pivot is zero. Typical matrices on echelon form are

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 4 & 2 \\ 0 & \boxed{7} & -1 & 5 \\ 0 & 0 & \boxed{3} & -1 \end{array}\right), \quad \left(\begin{array}{cccc|c} \boxed{1} & 2 & 4 & 7 & 10 \\ 0 & 0 & \boxed{3} & 3 & -6 \\ 0 & 0 & 0 & \boxed{4} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We have marked each pivot with a box. Notice that in Example 2.1-2.2, we stopped the process of elementary row operations once we had reached an echelon form.

A matrix has *reduced echelon form* when all pivots are 1, and all entries above and below a pivot are zero. The matrices given above are on echelon form, but not reduced echelon form. Typical examples of matrices on reduced echelon form are

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right), \quad \left(\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 0 & 10 \\ 0 & 0 & \boxed{1} & 0 & -6 \\ 0 & 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Notice the entries in the columns without pivots.

Proposition 2.1. *Any matrix is row-equivalent to a matrix on echelon form, but this echelon form is not unique, in general. Any matrix is row-equivalent to a unique reduced echelon form.*

Hence, starting from a given matrix, one might end up with several different echelon forms depending on the specific row operations that are used. However, the positions where the pivots end up are always the same, and are called the *pivot positions* of the matrix.

2.4 Gaussian elimination

Gaussian elimination is an efficient method for solving any linear system using systematic elimination of variables. It can be described in the following way:

- Write down the augmented matrix of the linear system.
- Use elementary row operations to reduced the matrix to an echelon form.
- Solve the linear system of the echelon form using back substitution.

Gauss-Jordan elimination is a variation of Gaussian elimination. The difference is that the elementary row operations in a Gauss-Jordan elimination continue until we reach the *reduced* echelon form of the matrix. This means a few extra row operations, but easier calculations in the final step since back substitution is now longer needed. There are also some theoretical advantages of Gauss-Jordan elimination since the reduced echelon form is unique.

Example 2.3. We use Gauss-Jordan elimination to solve the 3×3 linear system given by

$$\begin{array}{rcl} S \qquad\qquad + 0.05C &=& 5,000 \\ 0.40S + \qquad F + 0.40C &=& 40,000 \\ 0.10S + 0.10F + \qquad C &=& 10,000 \end{array} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0.40 & 1 & 0.40 & 40,000 \\ 0.10 & 0.10 & 1 & 10,000 \end{array}\right)$$

We start with the elementary row operations from Example 2.2, but continue with elementary row operations until we reach a reduced echelon form:

$$\begin{array}{ll}
\left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0.40 & 1 & 0.40 & 40,000 \\ 0.10 & 0.10 & 1 & 10,000 \end{array}\right) & \begin{array}{l} \\ R_2 \leftarrow R_2 + (-0.40)R_1 \\ R_3 \leftarrow R_3 + (-0.10)R_1 \end{array} \\
\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0 & 1 & 0.38 & 38,000 \\ 0 & 0.10 & 0.995 & 9,500 \end{array}\right) & \begin{array}{l} \\ \\ R_3 \leftarrow R_3 + (-0.10)R_2 \end{array} \\
\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0 & 1 & 0.38 & 38,000 \\ 0 & 0 & 0.957 & 5,700 \end{array}\right) & \begin{array}{l} \\ \\ R_3 \leftarrow R_3 \cdot (1/0.957) \end{array} \\
\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0 & 1 & 0.38 & 38,000 \\ 0 & 0 & 1 & 5,956 \end{array}\right) & \begin{array}{l} R_1 \leftarrow R_1 + (-0.05)R_3 \\ R_2 \leftarrow R_2 + (-0.38)R_3 \\ \\ \end{array} \\
\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4,702 \\ 0 & 1 & 0 & 35,737 \\ 0 & 0 & 1 & 5,956 \end{array}\right) &
\end{array}$$

We have found the reduced echelon form of the linear system, and can read off the solution of the system directly: $S = 4,702$ $F = 35,737$, $C = 5,956$.

2.5 Linear systems with no solutions

We say that a system is *consistent* if it has at least one solution, and *inconsistent* if it has no solutions. How do we recognize an inconsistent linear system when we use Gaussian elimination?

Proposition 2.2. *A linear system is inconsistent if and only if there is a pivot position in the last column of the augmented matrix.*

Example 2.4. Let us consider the linear system given by

$$\begin{aligned} x - y + z &= 4 \\ 3x + 2y - z &= 7 \\ -x - 4y + 3z &= 2 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 3 & 2 & -1 & 7 \\ -1 & -4 & 3 & 2 \end{array}\right)$$

We use Gaussian elimination to simplify the augmented matrix:

$$\begin{aligned} &\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 3 & 2 & -1 & 7 \\ -1 & -4 & 3 & 2 \end{array}\right) && \begin{array}{l} R_2 \leftarrow R_2 + (-3)R_1 \\ R_3 \leftarrow R_3 + (1)R_1 \end{array} \\ \Rightarrow &\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 5 & -4 & -5 \\ 0 & -5 & 4 & 6 \end{array}\right) && R_3 \leftarrow R_3 + (1)R_2 \\ \Rightarrow &\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 5 & -4 & -5 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{aligned}$$

The matrix has now echelon form, and there is a pivot in the last column. Therefore, the linear system has no solutions. In fact, the last equation of the linear system corresponding to the echelon form is

$$0x + 0y + 0z = 1$$

which clearly has no solutions.

2.6 Linear systems with many solutions

We know that some linear systems have (infinitely) many solutions. How do we recognize a linear system with infinitely many solutions when we use Gaussian elimination, and how do we find and describe the solutions?

Proposition 2.3. *A linear system has infinitely many solutions if and only if there is no pivot position in the last column of the augmented matrix, and there is at least one more column without a pivot position.*

In fact, when the last column (to the right of the vertical line) is not a pivot column, then the linear system is consistent and it is very useful to look at the pivot position in the remaining columns (to the left of the vertical line). Each of these columns corresponds to a variable. A variable is called a *basic* variable if it corresponds to a pivot column, and a *free* variable if it corresponds to a column without a pivot position.

It is always possible to express each basic variable in terms of the free variables, and it is easiest to do this using the reduced echelon form of the linear system. We say that there are d *degrees of freedom* when there are d free variables. If there is at least one degree of freedom, then there are infinitely many solutions.

Example 2.5. Let us consider the linear system given by

$$\begin{aligned} x + 2y + 3z &= 1 \\ 3x + 2y + z &= 1 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right)$$

We use Gaussian elimination to simplify the augmented matrix:

$$\begin{aligned} &\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right) && R_2 \leftarrow R_2 + (-3)R_1 \\ \Rightarrow &\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \end{array}\right) \end{aligned}$$

This is a matrix on echelon form, and the pivot columns are the first two columns. Hence the linear system has infinitely many solutions. In fact, the system has one degree of freedom, since x and y are basic variables while z is a free variable. To express x and y in terms of z, we continue with row operations until we find the reduced echelon form:

$$\begin{aligned} &\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \end{array}\right) && R_2 \leftarrow R_2 \cdot (-1/4) \\ \Rightarrow &\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0.5 \end{array}\right) && R_1 \leftarrow R_1 + (-2)R_2 \\ \Rightarrow &\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0.5 \end{array}\right) \end{aligned}$$

We use the reduced echelon form to express the basic variables x and y in terms of the free variable z:

$$\begin{aligned} x \quad - z &= 0 \\ y + 2z &= 0.5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= z \\ y &= 0.5 - 2z \end{aligned}$$

Notice that there is no single solution for this system. In fact, for any value of the free variable z, the equations above determine a unique value for the basic variables x and y. For instance, $z = 2$ gives the solution

$$x = 2,\ y = -3.5,\ z = 2$$

Since there are infinitely many choices for the free variable z, there are infinitely many solutions of the linear system.

Problems

2.1. Gaussian elimination I
Solve the following linear systems by Gaussian elimination:

$$a)\quad \begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned} \qquad\qquad b)\quad \begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x + 4y - 3z &= 0 \end{aligned}$$

2.2. Gauss-Jordan elimination
Solve the following linear system by Gauss-Jordan elimination:

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned}$$

2.3. Gaussian elimination II
Solve the following linear systems by Gaussian elimination:

$$a)\quad \begin{aligned} -4x + 6y + 4z &= 4 \\ 2x - y + z &= 1 \end{aligned} \qquad\qquad b)\quad \begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= 1 \end{aligned}$$

2.4. Variation of parameters
Discuss the number of solutions of the linear system

$$\begin{aligned} x + 2y + 3z &= 1 \\ -x + ay - 21z &= 2 \\ 3x + 7y + az &= b \end{aligned}$$

for all values of the parameters a and b.

2.5. Pivot positions
Find the pivot positions of the following matrix:

$$\begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 3 & 2 & 1 & 0 & 7 \\ -1 & 3 & 2 & 4 & 9 \end{pmatrix}$$

2.6. Basic and free variables
Show that the following linear system has infinitely many solutions, and determine the number of degrees of freedom:

$$\begin{aligned} x + 6y - 7z + 3w &= 1 \\ x + 9y - 6z + 4w &= 2 \\ x + 3y - 8z + 4w &= 5 \end{aligned}$$

Find free variables and express the basic variables in terms of the free ones.

2.7. Linear Systems
Solve the following linear systems by substitution and by Gaussian elimination:

$$a)\quad \begin{aligned} x - 3y + 6z &= -1 \\ 2x - 5y + 10z &= 0 \\ 3x - 8y + 17z &= 1 \end{aligned} \qquad\qquad b)\quad \begin{aligned} x + y + z &= 0 \\ 12x + 2y - 3z &= 5 \\ 3x + 4y + z &= -4 \end{aligned}$$

Chapter 3
The rank of a matrix

3.1 The rank of a matrix

Let A be any $m \times n$ matrix; that is, any rectangular table of numbers with m rows and n columns. The *rank* of A is the number of pivot positions in A, and we write $\mathrm{rk}A$ for the rank of A. There can not be more than one pivot position in a row or a column, hence $\mathrm{rk}A \leq m$ and $\mathrm{rk}A \leq n$.

Example 3.1. We compute the rank of the 3×3 matrix given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

The rank is equal to the number of pivot positions, so we compute an echelon form of A to find the pivot positions:

$$\begin{aligned}
&\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} && \begin{matrix} R_2 \leftarrow R_2 + (-1)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{matrix} \\
\Rightarrow &\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix} && R_3 \leftarrow R_3 + (1/2)R_2 \\
\Rightarrow &\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}
\end{aligned}$$

Since A has three pivot positions, we have $\mathrm{rk}A = 3$.

Notice that we can compute the rank of any matrix, if the matrix is the augmented matrix of a linear system or not. Also note that from the definition, the rank of the $m \times n$ zero matrix is

$$\operatorname{rk} \mathbf{0}_{m\times n} = \operatorname{rk} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 0$$

(since it has no pivot positions), and this is the only type of matrix with rank zero.

3.2 Rank and solutions of linear systems

The following proposition is a convenient way of describing the solutions of a linear system in terms of the pivot positions;

Proposition 3.1. *Let A be the coefficient matrix and $\hat{A}$ be the augmented matrix of a linear system in n variables.*

- *The linear system is consistent if* $\operatorname{rk} A = \operatorname{rk} \hat{A}$*, and inconsistent otherwise.*
- *Assume that the linear system is consistent. Then the linear system have a unique solution if* $\operatorname{rk} A = n$*, and infinitely many solutions if* $\operatorname{rk} A < n$*. In fact, the linear system has* $n - \operatorname{rk} A$ *degrees of freedom.*

The result is a reformulation of what we discovered in the previous chapter. In fact, we must have $\operatorname{rk} \hat{A} = \operatorname{rk} A$ or $\operatorname{rk} \hat{A} = \operatorname{rk} A + 1$. In the first case, all pivots of the augmented matrix $\hat{A}$ are also in A; that is, there is no pivot position in the last column of $\hat{A}$. Moreover, we must have $\operatorname{rk} A = n$ or $\operatorname{rk} A < n$, where n is the number of variables. The number of basic variables is equal to the number of pivot columns in A, or $\operatorname{rk} A$. The rest of the variables are free variables; hence there are $n - \operatorname{rk} A$ free variables.

3.3 Homogeneous linear systems

A *homogeneous* linear system is a linear system where all the constant terms are zero. The general $m \times n$ homogeneous linear system has the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{array}$$

where $a_{11}, a_{12}, \dots, a_{mn}$ are fixed numbers called parameters. Note that homogeneous linear systems always have the solution $(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$. This solution is called the *trivial solution*. Homogeneous linear systems are therefore consistent, and there are only two possibilities for its solutions:

- There is a unique solution.
- There are infinitely many solutions.

In the first case, the unique solution is of course the trivial solution. In the second case, there are (infinitely) many other solutions in addition to the trivial one, and these are called *non-trivial solutions*.

Proposition 3.2. *A homogeneous linear system in n variables with coefficient matrix A has only the trivial solution if $\operatorname{rk} A = n$, and it has non-trivial solutions if $\operatorname{rk} A < n$.*

Example 3.2. We solve the 3×3 homogeneous linear system given by

$$\begin{aligned} x + y + z &= 0 \\ x - y + z &= 0 \\ x + 2y + 4z &= 0 \end{aligned} \quad \Rightarrow \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

In Example 3.1, we computed the rank of the coefficient matrix A to be $\operatorname{rk} A = 3$. This means that $\operatorname{rk} A = n$ (since there are $n = 3$ variables), and the trivial solution $(x, y, z) = (0, 0, 0)$ is the only solution of this linear system.

Problems

3.1. Rank of a matrix I
Find the rank of the following matrices:

$$a) \quad \begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} \qquad c) \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix}$$

3.2. Rank of a matrix II
Find the rank of the following matrices:

$$(a) \quad \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix} \qquad c) \quad \begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix}$$

3.3. Homogeneous linear systems
Prove that any 4×6 homogeneous linear system has non-trivial solutions.

3.4. Linear system with parameters
Discuss the ranks of the coefficient matrix A and the augmented matrix $\hat{A}$ of the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 2q \\ 2x_1 - 3x_2 + 2x_3 &= 4q \\ 3x_1 - 2x_2 + px_3 &= q \end{aligned}$$

for all values of p and q. Use this to determine the number of solutions of the linear system for all values of p and q.

3.5. Midterm Exam in GRA6035 24/09/2010, Problem 3
Compute the rank of the matrix

$$A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & -2 & 4 \end{pmatrix}$$

3.6. Mock Midterm Exam in GRA6035 09/2010, Problem 3
Compute the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{pmatrix}$$

3.7. Midterm Exam in GRA6035 24/05/2011, Problem 3
Compute the rank of the matrix

$$A = \begin{pmatrix} 2 & 10 & 6 & 8 \\ 1 & 5 & 4 & 11 \\ 3 & 15 & 7 & -2 \end{pmatrix}$$

Solutions

Problems of Chapter 1

1.1 Net cost of charity donation

Without any charity donation, the state and federal tax would be given by the first two equations in Example 1.2 with $C = 0$:

$$\begin{aligned} S \quad &= \quad 5000 \\ 0.40S + F &= 40,000 \end{aligned}$$

We solve the first equation and get $S = 5,000$. Then we substitute this value for S in the second equation and get $F = 40,000 - 0.40(5,000) = 38,000$. Therefore, that state and federal tax i $S = 5,000$, $F = 38,000$. The profit net of taxes would in this case be

$$100,000 - 5,000 - 38,000 = 57,000$$

When a charity contribution is made as in Example 1.2, the profit net of taxes and donation are

$$100,000 - 4,207 - 35,737 - 5,956 = 53,605$$

The net profit is reduced by $3,395$ when the donation is made. Therefore, the net cost of the donation of \$$5,956$ is \$$3,395$.

1.2 From linear system to augmented matrix

The coefficient matrix and the augmented matrix of the system is given by

$$a) \quad \begin{pmatrix} 2 & 5 \\ 3 & -7 \end{pmatrix}, \quad \left(\begin{array}{cc|c} 2 & 5 & 6 \\ 3 & -7 & 4 \end{array}\right) \qquad b) \quad \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{pmatrix}, \quad \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ 1 & -2 & 4 & 3 \end{array}\right)$$

1.3 From augmented matrix to linear system

The linear system is given by

$$\begin{aligned} x + 2y \phantom{{}+z} &= 4 \\ 2x - 3y + z &= 0 \\ 7x + 4y + z &= 3 \end{aligned}$$

1.4 Solution by substitution

We solve the linear system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned}$$

by substitution. First, we solve the first equation for z and get $z = 1 - x - y$. Then we substitute this expression for z in the last two equations. We get

$$\begin{aligned} -2y &= 3 \\ -3x - 2y &= 3 \end{aligned}$$

We solve the first equation for y, and get $y = -1.5$. Then we substitute this value for y in the second equation, and get $x = 0$. Finally, we substitute both these values in $z = 1 - x - y$ and get $z = 2.5$. The solution is therefore $x = 0,\ y = -1.5,\ z = 2.5$.

1.5 Variation of parameters by substitution

We solve the linear system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + z &= h \end{aligned}$$

by substitution. First, we solve the first equation for z and get $z = 1 - x - y$. Then we substitute this expression for z in the last two equations. We get

$$\begin{aligned} -2y &= 3 \\ y &= h - 1 \end{aligned}$$

We solve the first equation for y, and get $y = -1.5$. Then we substitute this value for y in the second equation, and get $-1.5 = h - 1$. If $h = -0.5$, this holds and the system have solutions (x is a free variable, $y = -1.5$ and $z = 1 - x - y = 2.5 - x$). If $h \neq -0.5$, then this leads to a contradiction and the system have no solutions. Therefore, the linear system have solutions if and only if $h = -0.5$.

Problems of Chapter 2

General remark: In some of the problems, we compute an echelon form. Since the echelon form is not unique, it is possible to get to another echelon form than the one indicated in the solutions below. However, the pivot positions should be the same.

2.1 Gaussian elimination I
The linear systems have the following augmented matrices:

$$a)\quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 4 & 7 \end{array}\right) \qquad b)\quad \left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 1 & 1 & 1 & -2 \\ 2 & 4 & -3 & 0 \end{array}\right)$$

a) To solve the system, we reduce the system to an echelon form using elementary row operations. The row operations are indicated.

$$\begin{array}{ll} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 4 & 7 \end{array}\right) & \begin{array}{l} R_2 \leftarrow R_2 + (-1)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{array} \\ \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right) & R_3 \leftarrow R_3 + (0.5)R_2 \\ \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 3 & 7.5 \end{array}\right) & \end{array}$$

From the last equation we get $z = 2.5$, substitution in the second equation gives $y = -1.5$, and substitution in the first equation gives $x = 0$. Therefore, the solution of a) is $x = 0,\ y = -1.5,\ z = 2.5$.
b) To solve the system, we reduce the system to an echelon form using elementary row operations. The row operations are indicated.

$$\begin{array}{ll} \left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 1 & 1 & 1 & -2 \\ 2 & 4 & -3 & 0 \end{array}\right) & \begin{array}{l} R_2 \leftarrow R_2 + (-0.5)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{array} \\ \Rightarrow \left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 0 & 0 & 1.5 & -3 \\ 0 & 2 & -2 & -2 \end{array}\right) & \begin{array}{l} R_2 \leftarrow R_3 \\ R_3 \leftarrow R_2 \end{array} \\ \Rightarrow \left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 1.5 & -3 \end{array}\right) & \end{array}$$

From the last equation we get $z = -2$, substitution in the second equation gives $y = -3$, and substitution in the first equation gives $x = 3$. Therefore, the solution of b) is $x = 3,\ y = -3,\ z = -2$.

2.2 Gauss-Jordan elimination

We reduce the system to the reduced echelon form using elementary row operations:

$$
\begin{aligned}
&\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 4 & 7 \end{array}\right) && \begin{array}{r} R_2 \leftarrow R_2+(-1)R_1 \\ R_3 \leftarrow R_3+(-1)R_1 \end{array} \\
\Rightarrow &\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right) && R_3 \leftarrow R_3+(0.5)R_2 \\
\Rightarrow &\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 3 & 7.5 \end{array}\right) && \begin{array}{r} R_2 \leftarrow (-1/2)\cdot R_2 \\ R_3 \leftarrow (1/3)\cdot R_3 \end{array} \\
\Rightarrow &\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 1 & 2.5 \end{array}\right) && R_1 \leftarrow R_1+(-1)R_2+(-1)R_3 \\
\Rightarrow &\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 1 & 2.5 \end{array}\right)
\end{aligned}
$$

We read off the solution of the system: $x = 0,\ y = -1.5,\ z = 2.5$.

2.3 Gaussian elimination II

a) We reduce the linear system to an echelon form:

$$
\left(\begin{array}{ccc|c} -4 & 6 & 4 & 4 \\ 2 & -1 & 1 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} -4 & 6 & 4 & 4 \\ 0 & 2 & 3 & 3 \end{array}\right)
$$

We see that the system has infinitely many solutions (z is a free variable and x, y are basic variables). We reduce the system to a reduced echelon form:

$$
\left(\begin{array}{ccc|c} -4 & 6 & 4 & 4 \\ 0 & 2 & 3 & 3 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & -1.5 & -1 & -1 \\ 0 & 1 & 1.5 & 1.5 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 1.25 & 1.25 \\ 0 & 1 & 1.5 & 1.5 \end{array}\right)
$$

We see that $x + 1.25z = 1.25,\ y + 1.5z = 1.5$. Therefore the solution is given by $x = 1.25 - 1.25z,\ y = 1.5 - 1.5z$ (z is a free variable).

b) We reduce the linear system to an echelon form:

$$
\left(\begin{array}{cc|c} 6 & 1 & 7 \\ 3 & 1 & 4 \\ -6 & -2 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 6 & 1 & 7 \\ 0 & 0.5 & 0.5 \\ 0 & -1 & 8 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 6 & 1 & 7 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 9 \end{array}\right)
$$

We see that the system has no solutions.

2.4 Variation of parameters

We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & a+2 & -18 & 3 \\ 0 & 1 & a-9 & b-3 \end{array}\right)$$

We interchange the last two rows to avoid division with $a+2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & a+2 & -18 & 3 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & 0 & -18-(a-9)(a+2) & 3-(b-3)(a+2) \end{array}\right)$$

We compute $-18-(a-9)(a+2) = 7a-a^2$. So when $a \neq 0$ and $a \neq 7$, the system has a unique solution. When $a = 0$, we compute $3-(b-3)(a+2) = 9-2b$. So when $a = 0$ and $b \neq 9/2$, the system is inconsistent, and when $a = 0$, $b = 9/2$, the system has infinitely many solutions (one degree of freedom). When $a = 7$, we compute $3-(b-3)(a+2) = 30-9b$. So when $a = 7$ and $b \neq 30/9 = 10/3$, the system is inconsistent, and when $a = 7$, $b = 10/3$, the system has infinitely many solutions (one degree of freedom).

2.5 Pivot positions

We redude the matrix to an echelon form using row elementary row operations:

$$\begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 3 & 2 & 1 & 0 & 7 \\ -1 & 3 & 2 & 4 & 9 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 0 & -7 & -11 & -3 & -14 \\ 0 & 6 & 6 & 5 & 16 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 0 & -7 & -11 & -3 & -14 \\ 0 & 0 & -24/7 & * & * \end{pmatrix}$$

We have not computed the entries marked $*$ since they are not needed to find the pivot positions. The pivot positions in the matrix are marked with a box:

$$\begin{pmatrix} \boxed{1} & 3 & 4 & 1 & 7 \\ 3 & \boxed{2} & 1 & 0 & 7 \\ -1 & 3 & \boxed{2} & 4 & 9 \end{pmatrix}$$

2.6 Basic and free variables

We find the augmented matrix and reduce it to an echelon form using elementary row operations:

$$\left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & -3 & -1 & 1 & 4 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{array}\right)$$

We see that the system has infinitely many solutions and one degree of freedom (z is a free variable and x, y, w are basic variables). To express x, y, w in terms of z, we find the reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 5/2 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cccc|c} 1 & 0 & -9 & 0 & -7/2 \\ 0 & 1 & 1/3 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 5/2 \end{array}\right)$$

We see that $x - 9z = -7/2$, $y + z/3 = -1/2$ and $w = 5/2$. This means that the solution is given by $x = 9z - 7/2$, $y = -z/3 - 1/2$, $w = 5/2$ (z is a free variable).

2.7 Linear Systems

a) We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\left(\begin{array}{ccc|c} 1 & -3 & 6 & -1 \\ 2 & -5 & 10 & 0 \\ 3 & -8 & 17 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & -3 & 6 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Back substitution gives the solution $x = 5,\ y = 6,\ z = 2$.

b) We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 12 & 2 & -3 & 5 \\ 3 & 4 & 1 & -4 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -35 & -35 \end{array}\right)$$

Back substitution gives the solution $x = 1,\ y = -2,\ z = 1$.

Problems of Chapter 3

3.1 Rank of a matrix I

a) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

We see that the rank of A is 1 since there is one pivot position.

b) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 3 & 4 \\ 0 & -6 & -7 \end{pmatrix}$$

We see that the rank of A is 2 since there are two pivot positions.

c) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the rank of A is 2 since there are two pivot positions.

3.2 Rank of a matrix II

a) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & -4 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

We see that the rank of A is 3 by counting pivot positions.
b) We find an echelon form of the matrix:

$$\begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 3 & 7 \\ 0 & 4.5 & 4.5 & 4.5 \\ 0 & 0.5 & 0.5 & 0.5 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 3 & 7 \\ 0 & 4.5 & 4.5 & 4.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the rank of A is 2 by counting pivot positions.
c) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 5 & 3 & 0 \\ 0 & -1 & -2 & -2 \\ 0 & -9 & -4 & 2 \end{pmatrix}$$

We interchange the two middle rows to get easier computations:

$$\begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 5 & 3 & 0 \\ 0 & -9 & -4 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & -7 & -10 \\ 0 & 0 & 14 & 20 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & -7 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the rank of A is 3 by counting pivot positions. T

3.3 Homogeneous linear systems

Let A be the 4×6 coefficient matrix of the homogeneous linear system. Then $n = 6$ (there are 6 variables) while $\operatorname{rk} A \leq 4$ (there cannot be more than one pivot position in each row). So there are at least two degrees of freedom, and the system has non-trivial solutions.

3.4 Linear system with parameters

We find the coefficient matrix A and the augmented matrix $\hat{A}$ of the system:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & -2 & p \end{pmatrix}, \quad \hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right)$$

Then we compute an echelon form of $\hat{A}$ (which contains an echelon form of A as the first three columns):

$$\hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & p-3 & -5q \end{array}\right)$$

By counting pivot positions, we see that the ranks are given by

$$\operatorname{rk} A = \begin{cases} 3 & p \neq 3 \\ 2 & p = 3 \end{cases} \qquad \operatorname{rk} \hat{A} = \begin{cases} 3 & p \neq 3 \text{ or } q \neq 0 \\ 2 & p = 3 \text{ and } q = 0 \end{cases}$$

The linear system has one solution if $p \neq 3$, no solutions if $p = 3$ and $q \neq 0$, and infinitely many solutions (one degree of freedom) if $p = 3$ and $q = 0$.

3.5 Midterm Exam in GRA6035 24/09/2010, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & -2 & 4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & 7 \\ 0 & 0 & 0 & 0 & -6 \end{pmatrix}$$

Hence A has rank 3.

3.6 Mock Midterm Exam in GRA6035 09/2010, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence A has rank 3.

3.7 Midterm Exam in GRA6035 24/05/2011, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 2 & 10 & 6 & 8 \\ 1 & 5 & 4 & 11 \\ 3 & 15 & 7 & -2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 5 & 4 & 11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence A has rank 2.