Graded D-modules over Monomial Curves

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Contents

Preface

The work on this thesis started in early 1995. I had finished my Cand.Scient. degree in mathematics in late 1994, with professor O.A. Laudal as my supervisor. The subject for my master thesis had been connections on modules over affine singular curves.

It was a natural choice to continue the study of differential structures on affine singular curves when I first started working on this thesis. I have been fascinated by this theory ever since I was first introduced to it by my supervisor during my master studies.

I formally started my work on this thesis in September 1995, as a Research Fellow at the Department of Mathematics at the University of Oslo, with professor O.A. Laudal as my thesis supervisor. I would like to express my gratitude to him for introducing me to the subject of this thesis, and for his continuous advice and support through all these years. His enthusiasm never seems to end, and has been an inspiration for me.

I would like to thank all the people in the algebraic geometry group at the Department of Mathematics in Oslo for their valuable help on many occasions. In particular, I would like to thank Trond Stølen Gustavsen, Arvid Siqueland and Runar Ile for many interesting discussions.

An essential part of the research for this thesis was done while I visited professor Ph. Maisonobe at the Université de Nice in the period January - June 1997. This was a very fruitful time for me, and I wish to thank professor Maisonobe and all the people at the Université de Nice for their advice on my research and for providing excellent working conditions. I also wish to thank professor C. Simpson for his advice during my visit at the Université Paul Sabatier in Toulouse in the period January - April 1999.

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Part of the research described in this thesis is joint work with Henrik Vosegaard. I am indebted to him for his cooperation and his interest in my research. I am also thankful to professor A. Geramita for his expert advice on Hilbert functions.

Finally, I want to thank my family and my friends for their support. But most of all, I want to thank Elisabeta for her support, help and inspiration.

Notations and Conventions

A ring is always assumed to be an associative ring with 1, but not necessarily commutative. All ring homomorphisms are supposed to preserve 1, and a subring means a subring with (the same) unit. An ideal always means a two-sided ideal. All modules are assumed to be unitary. As far as possible, we try to use letters such as A, B, C to denote commutative rings, D, E, F to denote rings of differential operators, and R, S, T to denote other associative rings.

The letter k is reserved for a fixed, algebraically closed field of characteristic 0. A k-algebra R means a ring R with a structural ring homomorphism $k \to R$ such that the image of k is in the centre of R. We shall always identify k with its image under this map.

As usual, $k[x_1, \dots, x_n]$ denotes the polynomial ring in n commuting variables, and similarly $k[[x_1, \dots, x_n]]$ the ring of formal power-series in n commuting variables. We will use the notation $k\{x_1, \dots, x_n\}$ and $k\{\{x_1, \dots, x_n\}\}$ to denote their non-commutative counterparts: The free, associative k-algebra on n symbols and the ring of formal power-series in n non-commuting variables.

The opposite ring of a ring R is denoted R^{op}. All functors are covariant functors, unless the opposite is explicitly stated. For a family $\{V_{ij}\}_{1 \leq i,j \leq n}$ of k-vector spaces, we use matrix notation and write (V_{ij}) for the direct sum $(V_{ij}) = \oplus V_{ij}$.

We denote by ACC the ascending chain condition, and by DCC the descending chain condition. Let R be a ring. We say that a left or right R-module M is Artinian (Noetherian) if it satisfies the DCC (ACC) on its submodules. We say that the ring R is left Artinian (Noetherian) if it is Artinian (Noetherian) as left R-module, and that it is right Artinian (Noetherian) if it is Artinian (Noetherian) as right R-module. We say that R is Artinian (Noetherian) if it is both left and right Artinian (Noetherian).

We write $\mathbf{C}$ for the set of complex numbers, $\mathbf{Q}$ for the set of rationals, $\mathbf{Z}$ for the set of integers and $\mathbf{N_0}$ for the set of natural numbers including 0.

Introduction

In algebraic geometry, there is a well-developed theory of differential structures on smooth algebraic varieties in characteristic 0. This theory is, of course, strongly influenced by the corresponding analytic theory. It contains the theory of integrable connections and the theory of D-modules. The aim of this work is to contribute to the understanding of differential structures on singular algebraic varieties in characteristic 0.

Let k be an algebraically closed field of characteristic 0 and let A be a reduced, commutative k-algebra of finte type, corresponding to an affine algebraic variety $X = \operatorname{Spec} A$. We are interested in differential structures on X.

In particular, we consider the following special case: Let $\Gamma \subseteq \mathbf{N_0}$ be a numerical semigroup. That is, let Γ be a subset of $\mathbf{N_0}$ which contains 0 and is closed under addition, such that $\mathbf{N_0} \backslash \Gamma$ is a finite set. We let $A = k[\Gamma]$ denote the corresponding k-algebra, which is defined as the k-subalgebra of $k[t]$ with k-linear basis $\{t^\gamma : \gamma \in \Gamma\}$. The corresponding affine algebraic variety $X_\Gamma = \operatorname{Spec} A$ is a curve, and we call X_Γ the monomial curve corresponding to Γ.

The definition of the ring of k-linear differential operators on a smooth algebraic variety X over k was generalized by Grothendieck. Grothendieck's definition appeared in full generality in Grothendieck [**15**], and we shall refer to it in the special case when X is an affine algebraic variety $X = \operatorname{Spec} A$ over k: We let $D = \mathrm{D}(X) = \mathrm{D}(A)$ be the k-subalgebra of $\operatorname{End}_k(A)$ given by

$$\mathrm{D}(A) = \bigcup_{p \in \mathbf{Z}} \mathrm{D}^p(A),$$

where $\mathrm{D}^p(A)$ is the k-linear subspace of $\operatorname{End}_k(A)$ given in the following way: When $p < 0$, $\mathrm{D}^p(A) = 0$. and when $p \geq 0$, $\mathrm{D}^p(A)$ consists of the k-linear endomorphisms $P \in \operatorname{End}_k(A)$ such that the commutator $[P, a] \in \mathrm{D}^{p-1}(A)$ for all $a \in A$. In particular, $\mathrm{D}(A)$ is an associative subring of $\operatorname{End}_k(A)$, and we call it the ring of k-linear differential operators on A (or X).

In general, we have $\mathrm{D}^0(A) = A$, where the elements of A are considered as k-linear operators on A by left multiplication, and $\mathrm{D}^1(A) = A \oplus \operatorname{Der}_k(A)$. We denote by $\Delta(A)$ the subring of $\mathrm{D}(A)$ generated by the differential operators in $\mathrm{D}^1(A)$. It is well-known that if A is a regular k-algebra, or equivalently if X is a smooth variety, then $\Delta(A) = \mathrm{D}(A)$. It is believed that regularity is essential for this to happen, and Nakai's conjecture states that if A is an integral domain, then $\Delta(A) = \mathrm{D}(A)$ if and only if A is regular. Nakai's conjecture holds if X is a curve, see Mount and Villamajor [**25**]. For higher dimensions, the problem is still open.

Let X be an affine algebraic variety, and let A be the affine coordinate ring of X. We consider the following problems:

A: Let $D = D(X)$ the the ring of diffential operators on X, and denote by a D-module any left D-module which is finitely generated. Classify all D-modules with certain given properties.

B: A module with covariant derivative is a couple (M, ∇), where M is an A-module and $\nabla : \operatorname{Der}_k(A) \to \operatorname{End}_k(M)$ is an A-linear map such that

$\nabla_D(am) = a\nabla_D(m) + D(a)m$ for all $D \in \mathrm{Der}_k(A),\ a \in A,\ m \in M$. We say that the covariant derivative ∇ is integrable or flat if ∇ is a homomorphism of k-Lie algebras. Classify all couples (M, ∇) of modules with integrable covariant derivative with certain given properties.

When X is smooth, problem **A** and problem **B** are equivalent. However, they are different in an essential way when X is singular, and in this thesis, we shall consider problem **A** and **B** for a monomial curve X_Γ.

Let us first consider problem **A.** It is well-known that the ring $D = \mathrm{D}(A)$ is not well-behaved when A is the affine coordinate ring of a general affine variety. In fact, it was shown in Bernstein, Gelfand and Gelfand [**2**] that when X is the cubic cone

$$X = V(x^3 + y^3 + z^3) \subseteq \mathbf{C}^3,$$

then $D = \mathrm{D}(X)$ is not a finitely generated k-algebra, and D is neither left nor right Noetherian. However, in the case of curves, the situation is much better. The following theorems were shown in Smith and Stafford [**32**]:

Theorem 0.1 (Smith, Stafford)**.** *Let X be an irreducible affine algebraic variety over an algebraically closed field k of characteristic 0, and let A be the affine coordinate ring of X. Then the following conditions are equivalent:*
i) The normalization map $\overline{X} \to X$ is injective.
ii) $\mathrm{D}(A)$ is a simple ring.
iii) A is a simple left $\mathrm{D}(A)$-module.
iv) $\mathrm{D}(A)$ is Morita-equivalent to $\mathrm{D}(\overline{A})$.

Theorem 0.2 (Smith, Stafford)**.** *Let X be an irreducible affine algebraic variety over an algebraically closed field k of characteristic 0, and let A be the affine coordinate ring of X. If the normalization map $\overline{X} \to X$ is injective, then $\mathrm{D}(A)$ has the following properties:*
i) $\mathrm{D}(A)$ is a finitely generated k-algebra.
ii) $\mathrm{D}(A)$ is a Noetherian ring.
iii) $\mathrm{D}(A)$ has Krull dimension 1 and Gelfand-Kirillov dimension 2.
iv) $\mathrm{D}(A)$ is an hereditary ring.

A main technique used by Smith and Stafford was to compare the rings of differential operators $\mathrm{D}(X)$ and $\mathrm{D}(\overline{X})$. When X is an affine variety of higher dimension, these rings are related, and this is useful when $\overline{X}$ is smooth. But for curves, the relationship between $\mathrm{D}(X)$ and $\mathrm{D}(\overline{X})$ is very close, and this is the idea behind much of the work on the ring of differential operators on curves. In the special case of monomial curves, we see that $\overline{X} = \mathbf{A}^1$ and the normalization map is a bijection. So for any monomial curve X_Γ, there are very strong structural results on the ring $D = \mathrm{D}(X)$. Moreover, $\mathrm{D}(\overline{X}) = \mathrm{A}_1(k)$, the first Weyl algebra. This is the associative k-algebra generated by x and ∂, which has the relation $\partial x - x\partial = 1$. A lot of consideration has been given to this ring, in particular in Dixmier [**11, 12**], and Block [**4**].

In a joint work with Henrik Vosegaard, we have obtained a very explicit description of the ring structure of $D = \mathrm{D}(X)$ when X is a monomial curve. The ring $\mathrm{D}(X)$ possesses a graded structure, inherited from the graded structure of the affine coordinate ring A of X, and our description of D relies on this graded structure. It has recently been brought to our attention that some of our results had already been obtained by Jones in his PhD thesis Jones [**17**].

We develop some new results on Hilbert functions of graded modules over graded k-algebras not necessarily generated in degree 1. Using these results, we are able to define the dimension and multiplicity of D-modules over monomial curves.

We show that Bernstein's equation $d(M) \geq 1$ holds for all non-zero D-modules M, where $d(M)$ is the dimension of M, and we define a holonomic D-module M to be a D-module M such that $M = 0$ or such that $M \neq 0$ and $d(M) = 1$. Furthermore, we are able to prove that the category of holonomic D-modules and the category of Artinian D-modules coincide when $D = \mathrm{D}(X)$ for a monomial curve X. The proof is similar to the proof of the corresponding result for the Weyl algebra $\mathrm{A}_1(k)$.

The main result in the direction of problem **A**, is the classification of the graded, holonomic D-modules which are indecomposable. We have obtained the following result (see below for the notation used in the theorem):

Theorem 0.3. *Let X be a monomial curve, and let $D = \mathrm{D}(X)$ be the ring of differential operators on X. The set of equivalence classes of graded, holonomic D-modules which are indecomposable, up to graded isomorphisms of degree* 0 *and twists, are given by*

$$\{M(\alpha, n) : \alpha \in I^*, n \geq 1\} \cup \{M_A(n) : n \geq 1\} \cup \{M_B(n) : n \geq 1\}.$$

where $I \subseteq k$ is a subset containing 0 *such that the natural quotient map $I \to k/\mathbf{Z}$ is a bijection, and $I^* = I \setminus \{0\}$.*

Assume that $X = \mathbf{A}^1$ and let $D = \mathrm{A}_1(k)$. Then the indecomposable, graded, holonomic D-modules are explicitly given in the following way: For all $\alpha \in I^*$ and $n \geq 1$, we have $M(\alpha, n) = D/D(E - \alpha)^n$. If $n \geq 1$ is an even number with $n = 2m$, then $M_A(n) = D/D(t\partial)^m$ and $M_B(n) = D/D(\partial t)^m$. If $n \geq 1$ is an odd number with $n = 2m + 1$, then $M_A(n) = D/D\partial(t\partial)^m$ and $M_B(n) = D/Dt(\partial t)^m$. In the general case, when X is a monomial curve, the graded, holonomic, indecomposable D-modules over X are given by the Morita equivalence between $\mathrm{D}(X)$ and $\mathrm{A}_1(k)$.

Let D' be the corresponding ring of differential operators in the local analytic category. More explicitly, let D' be given as $D' = C[\partial]$, where C is the ring of germs of analytic functions on $\mathbf{C}$. There is a classification of indecomposable, holonomic D'-modules with regular singularities, see Briançon and Maisonobe [**7**]. We observe that the indecomposable, graded, holonomic D-modules over a monomial curve correspond exactly to the indecomposable, holonomic D'-modules with regular singularities in the local analytic category.

Let $\mathbf{M} = \{M_1, \dots, M_p\}$ be a finite family of non-isomorphic D-modules. We define an extension of extensions of the family $\mathbf{M}$ to be a couple (M, C), where M is a D-module, and $C = (C_i)$ is a co-filtration of M of length n such that

$$\ker(C_i \to C_{i-1}) \cong M_{l_i}$$

for $1 \leq i \leq n$. Let G be the ordered, directed graph associated with (M, C) in the following way: Let the nodes of G be $N = \{1, 2, \dots, p\}$, and let the edges of G be $E = \{a_1, \dots, a_{n-1}\}$, where a_i is an edge from node l_i to node l_{i+1} for $1 \leq i \leq n-1$, and where the total order of E is given by $a_1 < \cdots < a_{n-1}$. The graph G is called the extension type of (M, C), and we say that a D-module M has extension type G if there exists a co-filtration C such that (M, C) is an extension of extensions of $\mathbf{M}$ with extension type G.

There is a non-commutative deformation theory, due to Laudal, which is described in Laudal [**21, 22**]. Using this deformation theory, it is possible to construct a family $M(\mathbf{M}, \mathrm{G})$ of D-modules containing all D-modules which are extensions of extensions of the family $\mathbf{M}$ with extension type G. We have used this construction to show the above classification result when $X = \mathbf{A}^1$. In the general case, when $D = \mathrm{D}(X)$ for a monomial curve X, we have used the Morita equivalence between D and $\mathrm{A}_1(k)$ to complete the classification.

In the direction of problem **B**, let X be a Gorenstein monomial curve. We have shown the following existence result for modules with integrable covariant derivative on X:

Theorem 0.4. *Let X be a Gorenstein monomial curve, and let A be the affine coordinate ring of X. For any graded, torsion free A-module M of rank 1, there exists an integrable covariant derivative ∇ on M.*

In **Chapter 1**, we give the basic definitions of rings of differential operators, following Grothendieck. We also mention some elementary properties of rings of differential operators for later reference. Most of these results are well-known in the literature.

In **Chapter 2**, we study the ring of differential operators on monomial curves. We obtain a very explicit description of the ring of differential operators, the associated graded ring, and the module of derivations. We also obtain information about generators in all these cases, and for the associated graded ring and the module of derivations, we find minimal generating set. The work described in this chapter is joint work with Henrik Vosegaard. Some of the results in this chapter can also be found in Jones [**17**].

In **Chapter 3**, we study modules over the ring $D = \mathrm{D}(X)$ of differential operators on a monomial curve. We show that any D-module has a dimension and a multiplicity. We generalize classical results on Hilbert functions on graded modules in order to make these definitions. We also generalize some localization procedures used in Block [**4**] to prepare the classification of all simple, graded D-modules.

In **Chapter 4**, we study modules with connection. We define modules with connections, modules with covariant derivative and modules with $\mathbf{g}$-connections for a Lie-Cartan pair $(A, \mathbf{g})$. We also develop the obstruction theory for connections and $\mathbf{g}$-connections on a given A-module M, following Laudal. Finally, we show that any graded, torsion-free A-module M of rank 1 over a monomial curve has an integrable covariant derivative.

In **Chapter 5**, we give an introduction to a non-commutative deformation theory, following Laudal. In particular, we define a deformation functor $\mathrm{Def}_{\mathbf{M}}$ on the category $\mathbf{a}_p$ of p-pointed Artinian k-algebras, which describes the simultaneous deformations of a finite family of modules. We also show that this functor has a pro-representing hull $H(\mathbf{M})$, and we construct this hull using the obstruction morphism $o : \mathrm{T}^2 \to \mathrm{T}^1$. In the last section, we use non-commutative deformation theory to construct a family $M(\mathbf{M}, \mathrm{G})$ of D-modules which contains all extensions of extensions of the finite family $\mathbf{M}$ with fixed extension type G.

In **Chapter 6**, we study graded, holonomic modules over the Weyl algebra $\mathrm{A}_1(k)$. First, we use the results from chapter 3 to classify all simple, graded $\mathrm{A}_1(k)$-modules. Then, we use methods from chapter 5 to classify all graded, holonomic and indecomposable $\mathrm{A}_1(k)$-modules.

In **Chapter 7**, we study graded, holonomic D-modules over a monomial curve X. We use the Morita equivalence betweeen D and $\mathrm{A}_1(k)$ to classify all graded, holonomic and indecomposable D-modules.

In **Appendix A**, we describe Hochschild cohomology. We also show the relationship between Hochschild cohomology and Ext groups.

CHAPTER 1

Rings of differential operators

In this chapter, we shall define the filtered k-algebra $\mathrm{D}(A)$ of k-linear differential operators on a commutative k-algebra A, following Grothendieck [**15**]. We shall also recall some elementary properties of the ring of differential operators $\mathrm{D}(A)$ and its associated graded ring $\operatorname{gr}\mathrm{D}(A)$. Although most of these results are well-known, there are some which, as far as we know, do not appear in the literature. We remind the reader that in this chapter, all rings denoted A, B, C are assumed to be commutative.

Our main references for this chapter are the following: Grothendieck [**15**], Másson [**23**], Smith and Stafford [**32**] and Muhasky [**26**].

1. Basic definitions

Let A be a commutative k-algebra. We denote by $\operatorname{End}_k(A)$ the vector space of k-linear endomorphisms of A, and consider each $\phi \in \operatorname{End}_k(A)$ as a left operator on A. The k-vector space $\operatorname{End}_k(A)$ has an associative k-algebra structure, with multiplication given by composition of left operators. Furthermore, there is a left operator $L_a \in \operatorname{End}_k(A)$ for all $a \in A$, given by left multiplication by a, and $a \mapsto L_a$ induces an inclusion of k-algebras $A \to \operatorname{End}_k(A)$. We shall always identify A with its image under this inclusion.

We denote by $\operatorname{Der}_k(A)$ the set of endomorphisms $\phi \in \operatorname{End}_k(A)$ which fulfill the Leibniz rule $\phi(ab) = a\phi(b) + \phi(a)b$ for all $a, b \in A$. It is clear that $\operatorname{Der}_k(A)$ is a left A-submodule of $\operatorname{End}_k(A)$. Let $[\phi, \psi] = \phi\psi - \psi\phi$ for all $\phi, \psi \in \operatorname{End}_k(A)$ denote the Lie product in $\operatorname{End}_k(A)$. Then we see that $\operatorname{Der}_k(A)$ is closed under the Lie product.

Following Grothendieck [**15**], we let $\mathrm{D}(A) \subseteq \operatorname{End}_k(A)$ be the subring of left operators on A given in the following way: We define $\mathrm{D}^p(A) = 0$ for $p < 0$, and recursively define

$$\mathrm{D}^p(A) = \{\phi \in \operatorname{End}_k(A) : [\phi, a] \in \mathrm{D}^{p-1}(A) \text{ for all } a \in A\}$$

for $p \geq 0$. It follows that $\mathrm{D}^p(A)$ is a k-linear subspace of $\operatorname{End}_k(A)$ for all $p \geq 0$. The equality $\mathrm{D}^0(A) = A$ is clear. Furthermore, the decomposition $P = P(1) + (P - P(1))$ gives $\mathrm{D}^1(A) = A \oplus \operatorname{Der}_k(A)$, and an induction argument gives

$$\mathrm{D}^p(A)\,\mathrm{D}^q(A) \subseteq \mathrm{D}^{p+q}(A)$$

for all integers p, q. We define $\mathrm{D}(A) = \cup\, \mathrm{D}^p(A)$, which is an associative k-subalgebra of $\operatorname{End}_k(A)$ by the previous remark. We refer to this ring as the *ring of differential operators* on A, and we refer to operators $P \in \mathrm{D}(A)$ as *differential operators*.

Notice that the expression $P(a)$ has several possible interpretations for a differential operator $P \in \mathrm{D}(A)$ and an element $a \in A$: Either as the left action of P on a, or as the composition of P and a considered as differential operators. We shall therefore write $P * a$ for the left action of $P \in \mathrm{D}(A)$ on $a \in A$, and $P(a)$ or Pa for the composition of these considered as differential operators. Using this notation, we write the decomposition given above as $P = P * 1 + (P - P * 1)$ for $P \in \mathrm{D}^1(A)$.

For all integers p, let $\operatorname{Der}^p(A)$ denote the set of differential operators $P \in \mathrm{D}^p(A)$ such that $P * 1 = 0$. Then, $\operatorname{Der}^p(A) \subseteq \mathrm{D}^p(A)$ is k-linear subspace consisting of the *high order derivations* of order p in the sense of Nakai [**28**]. It is clear that $\mathrm{Der}^p(A) = 0$ for $p \leq 0$ and that $\mathrm{Der}^1(A) = \mathrm{Der}_k(A)$. Furthermore, we have that $\mathrm{D}^p(A) = A \oplus \mathrm{Der}^p(A)$ for all integers $p \geq 0$: For any differential operator $P \in \mathrm{D}^p(A)$, we have $P = P * 1 + (P - P * 1)$ with $P * 1 \in A$, $P - P * 1 \in \mathrm{Der}^p(A)$, and we easily check that $A \cap \mathrm{Der}^p(A) = 0$ in $\mathrm{D}^p(A)$.

From the definition, we see that $\mathrm{D}^p(A) \subseteq \mathrm{D}^{p+1}(A)$ for all integers p. So the subspaces $\mathrm{D}^p(A)$ form an ascending filtration of $\mathrm{D}(A)$, called *the order filtration*, and $\mathrm{D}(A)$ is a filtered k-algebra. We say that a differential operator $P \in \mathrm{D}(A)$ has *order* p if $P \in \mathrm{D}^p(A)$, but $P \notin \mathrm{D}^{p-1}(A)$. So every differential operator $P \neq 0$ has a uniquely defined order $d(P) = p$, which is a non-negative integer. By convention, $d(P) = -\infty$ if $P = 0$.

Lemma 1.1. *For all differential operators $P, Q \in \mathrm{D}(A)$, we have*
i) $d(P + Q) \leq \max\{d(P), d(Q)\}$,
ii) $d(PQ) \leq d(P) + d(Q)$,
iii) $d([P, Q]) \leq d(P) + d(Q) - 1$.

Proof. The first 2 inequalities are clear. For the third, the Jacobi identity of the Lie product gives the equation

$$[[P, Q], a] = [P, [Q, a]] + [[P, a], Q]$$

for all $a \in A$. So by induction on $n = d(P) + d(Q)$, the result follows, the case $n = 0$ (and indeed $n < 0$) being trivial. □

Let $\mathrm{gr}\,\mathrm{D}(A)$ denote the *associated graded* k-algebra associated with the order filtration of $\mathrm{D}(A)$. That is,

$$\mathrm{gr}\,\mathrm{D}(A) = \bigoplus_p \mathrm{D}^p(A)/\,\mathrm{D}^{p-1}(A).$$

This is a $\mathbf{Z}$-graded k-algebra, and by the last part of lemma 1.1, it is commutative. We shall denote the graded component of degree p by $\mathrm{gr}^p\,\mathrm{D}(A) = \mathrm{D}^p(A)/\,\mathrm{D}^{p-1}(A)$. Clearly, $\mathrm{gr}^0\,\mathrm{D}(A) = A$, and $\mathrm{gr}\,\mathrm{D}(A)$ is positively graded. In section 5, we shall see that there is a stronger version of lemma 1.1, which will describe the structure of $\mathrm{gr}\,\mathrm{D}(A)$ in more detail. This result is given in proposition 1.13.

2. Principal parts and further definitions

Let $\mu : A \otimes_k A \to A$ denote the k-linear multiplication map $\mu(a \otimes b) = ab$ for $a, b \in A$. It is a surjective morphism of k-algebras, and we denote by $J \subseteq A \otimes_k A$ the kernel of this morphism. Let $j_1, j_2 : A \to A \otimes_k A$ be the natural sections of μ, given by $j_1(a) = 1 \otimes a$ and $j_2(a) = a \otimes 1$ for $a \in A$.

We define the p'th *principal part* of A over k to be the k-algebra given by $\mathcal{P}^p(A/k) = (A \otimes_k A)/J^{p+1}$ for all $p \geq 0$. Notice that there are two natural k-algebra homomorphisms from A to $\mathcal{P}^p(A/k)$: These are the compositions of j_1 and j_2 with the canonical quotient map $A \otimes_k A \to \mathcal{P}^p(A/k)$. We shall always consider $\mathcal{P}^p(A/k)$ as an A-module via the morphism $A \to \mathcal{P}^p(A/k)$ induced by j_2, and we shall denote by $d^p(A/k) : A \to \mathcal{P}^p(A/k)$ the morphism induced by j_1. Notice that this morphism is not A-linear, but only k-linear.

Proposition 1.2. *Let $P \in \mathrm{End}_k(A)$ be an endomorphism, and $p \geq 0$ some integer. Then $P \in \mathrm{D}^p(A)$ if and only if there is an A-linear map $P' : \mathcal{P}^p(A/k) \to A$ such that $P' \circ d^p(A/k) = P$. In particular, there is an isomorphism of k-vector spaces $\mathrm{Hom}_A(\mathcal{P}^p(A/k), A) \to \mathrm{D}^p(A)$ induced by $d^p(A/k)$.*

Proof. See Grothendieck [**15**], proposition 16.8.8. □

Let M, N be left A-modules, and consider $\mathrm{Hom}_k(M,N)$ as an A-A bimodule in the natural way. Then we may form the bracket $[\phi, a] = \phi a - a\phi \in \mathrm{Hom}_k(M,N)$ for all $\phi \in \mathrm{Hom}_k(M,N),\ a \in A$. We define the differential operators from M to N in the following way: Let $\mathrm{D}^p(M,N) = 0$ for $p < 0$, and let $\mathrm{D}^p(M,N)$ be given recursively by

$$\mathrm{D}^p(M,N) = \{\phi \in \mathrm{Hom}_k(M,N) : [\phi, a] \in \mathrm{D}^{p-1}(M,N) \text{ for all } a \in A\}$$

for $p \geq 0$. Clearly, $\mathrm{D}^p(M,N)$ are k-linear subspaces of $\mathrm{Hom}_k(M,N)$ for all integers p, and by an induction argument, the composition of homomorphisms is such that

$$\mathrm{D}^q(N,P)\,\mathrm{D}^p(M,N) \subseteq \mathrm{D}^{p+q}(M,P)$$

for all left A-modules M, N, P and all integers p, q.

We define $\mathrm{D}(M,N) = \cup\,\mathrm{D}^p(M,N)$ for all left A-modules M, N. We refer to it as the *module of differential operators* from M to N, and we refer to operators $P \in \mathrm{D}(M,N)$ as *differential operators* from M to N. Notice that the notion of a differential operator from M to N does depend on the ring A, even if this is suppressed from the notation. We shall sometimes write $\mathrm{D}_A(M,N)$ for $\mathrm{D}(M,N)$ to emphasize that M, N are considered as left A-modules, and similarly $\mathrm{D}^p_A(M,N)$ for $\mathrm{D}^p(M,N)$.

Let $f : A \to B$ be a homomorphism of k-algebras, and let M, N be left B-modules. Then M, N are left A-modules via f, in the sense that $am = f(a)m$ and $an = f(a)n$ for all $a \in A,\ m \in M,\ n \in N$. An induction argument shows that there is an inclusion of Abelian groups $\mathrm{D}_B(M,N) \subseteq \mathrm{D}_A(M,N)$, and equality holds if f is surjective.

For simplicity, we shall write $\mathrm{D}(M)$ for $\mathrm{D}(M,M)$. It is clear that $\mathrm{D}(M)$ is a k-algebra in $\mathrm{End}_k(M)$. Furthermore, the k-linear subspaces $\mathrm{D}^p(M) = \mathrm{D}^p(M,M)$ form an ascending filtration of $\mathrm{D}(M)$, called the *order filtration*, and $\mathrm{D}(M)$ is a filtered k-algebra. We also see that $\mathrm{D}(M,N)$ is a $\mathrm{D}(N)$-$\mathrm{D}(M)$ bimodule, and that the linear subspaces $\mathrm{D}^p(M,N)$ form an ascending filtration of $\mathrm{D}(M,N)$, also called the *order filtration*. Consequently, $\mathrm{D}(M,N)$ is a filtered bimodule.

We say that a differential operator $P \in \mathrm{D}(M,N)$ has *order p* if $P \in \mathrm{D}^p(M,N)$ but $P \notin \mathrm{D}^{p-1}(M,N)$. So every non-zero differential operator $P \in \mathrm{D}(M,N)$ has a uniquely defined order $d(P) = p$, which is a non-negative integer. By convention, $d(P) = -\infty$ if $P = 0$. We also see that $\mathrm{D}^0(M,N) = \mathrm{Hom}_A(M,N)$, and that $\mathrm{D}(M,N) = 0$ if and only if $\mathrm{Hom}_A(M,N) = 0$.

Proposition 1.3. *Let M, N be left A-modules, $P \in \mathrm{Hom}_k(M,N)$ be a homomorphism, and $p \geq 0$ an integer. Then $P \in \mathrm{D}^p(M,N)$ if and only if there is an A-linear map $P' : \mathcal{P}^p(A/k) \otimes_A M \to N$ such that $P' \circ (d^p(A/k) \otimes_A \mathrm{id}_M) = P$. In particular, there is an isomorphism of k-vector spaces $\mathrm{Hom}_A(\mathcal{P}^p(A/k) \otimes_A M, N) \to \mathrm{D}^p(M,N)$ induced by $d^p(A/k)$.*

Proof. See Grothendieck [**15**], proposition 16.8.8. □

We end this section with some remarks: First, assume that $N \subseteq M$ is an inclusion of left A-modules. Then we have that

$$\mathrm{D}(M,N) = \{P \in \mathrm{D}(M) : P * M \subseteq N\}.$$

In particular, we see that $\mathrm{D}(A,K) = \{P \in \mathrm{D}(A) : P * A \subseteq K\}$ for all ideals $K \subseteq A$. Secondly, notice that for all left A-modules M, N, there are functors

$$\mathrm{D}(-,N) : A\text{-}\mathrm{Mod} \to \mathrm{D}(N)\text{-}\mathrm{Mod},$$
$$\mathrm{D}(M,-) : A\text{-}\mathrm{Mod} \to \mathrm{Mod}\text{-}\mathrm{D}(M),$$

the first contravariant, the second covariant. These functors are left exact. Furthermore, their images are in the corresponding filtered module categories.

3. Localization of algebras

Let A be a commutative k-algebra, and let $S \subseteq A$ be a multiplicatively closed subset. Consider the k-algebra homomorphism $f : A \to S^{-1}A$, where f is given by localization. There is a unique extension of differential operators in $\mathrm{D}(A)$ to differential operators in $\mathrm{D}(S^{-1}A)$. Explicit formulas for this were first given by Hart [**16**] for the case of integral domains, and then by Muhasky [**26**] for the general case.

Let us denote by $\mathrm{ad} : \mathrm{D}(A) \to \mathrm{End}_k(\mathrm{D}(A))$ the k-linear operator given by the equation

$$\mathrm{ad}(P)(Q) = [P, Q]$$

for all $P, Q \in \mathrm{D}(A)$. Using this notation, with the convention that $\mathrm{ad}(P)^0 = \mathrm{id}$, we obtain the following result:

Proposition 1.4. *Let $P \in \mathrm{D}(A)$ be a differential operator of order $d(P) = p \geq 0$. Then, for all $a \in A,\ s \in S$, the formula*

$$\overline{P} * (a/s) = \sum_{i=0}^{p} \mathrm{ad}(s)^i(P) * (a)/s^{i+1}$$

defines a differential operator $\overline{P} \in \mathrm{D}^p(S^{-1}A)$, such that $\overline{P} * f(a) = f(P * a)$ for all $a \in A$. Furthermore, the map $\mathrm{D}(f) : \mathrm{D}(A) \to \mathrm{D}(S^{-1}A)$ given by $P \mapsto \overline{P}$ is a homomorphism of filtered k-algebras, and its restriction to $A \subseteq \mathrm{D}(A)$ is the localization map $f : A \to S^{-1}A$.

Proof. See Muhasky [**26**], lemma 1.5 and 1.6. □

In his thesis, Másson [**23**] gave the following generalization of this result: Let $f : A \to B$ be a formally étale homomorphism of k-algebras. Then f induces a homomorphism of filtered k-algebras $\mathrm{D}(f) : \mathrm{D}(A) \to \mathrm{D}(B)$, and this homomorphism is characterized by the property that

$$\mathrm{D}(f)(P) * f(a) = f(P * a)$$

for all differential operators $P \in \mathrm{D}(A)$ and all elements $a \in A$. See Másson [**23**], theorem 2.2.5 for a proof of this generalization.

We recall the definition of *non-commutative localization*: Let R be an associative ring, and $S \subseteq R$ a multiplicatively closed subset. A left ring of fractions of R with respect to S is a ring R' together with a ring homomorphism $\phi : R \to R'$ such that the following conditions hold:

1. For all $s \in S$, $\phi(s)$ is invertible in R'.
2. Every element of R' is of the form $\phi(s)^{-1}\phi(r)$ with $r \in R,\ s \in S$.
3. We have $\phi(r) = 0$ if and only if $sr = 0$ for some $s \in S$.

We define a right ring of fractions similarly. Notice that if there is a (left or right) ring of fractions, it is unique up to unique isomorphism. We say that R' is a ring of fractions of R with respect to S if it is a left and right ring of fractions of R with respect to S. In this case, we shall denote R' by $S^{-1}R$.

Proposition 1.5. *Let A be a commutative k-algebra of finite type, $S \subseteq A$ be a multiplicatively closed subset, and $f : A \to S^{-1}A$ be the localization map. Then the k-algebra homomorphism $\mathrm{D}(f) : \mathrm{D}(A) \to \mathrm{D}(S^{-1}A)$ is a ring of fractions of $\mathrm{D}(A)$ with respect to S. In particular, there is an isomorphism of $S^{-1}A$-$\mathrm{D}(A)$ bimodules*

$$S^{-1}A \otimes_A \mathrm{D}(A) \to \mathrm{D}(S^{-1}A).$$

Proof. See Muhasky [**26**], proposition 1.9, or Másson [**23**], theorem 2.2.10. Notice that the latter proof also holds if $f : A \to B$ is any formally étale homomorphism of commutative k-algebras and $\mathcal{P}^p(A/k)$ is finitely presented as left A-module for all integers p. $\square$

We remark that the localization map f is injective if and only if S does not contain any zero-divisors. Furthermore, f is injective if and only if the k-algebra homomorphism $\mathrm{D}(f) : \mathrm{D}(A) \to \mathrm{D}(S^{-1}A)$ is injective. This follows from Másson [**23**], corollary 2.2.6, which in fact shows that a similar result holds for any formally étale homomorphism of commutative k-algebras $f : A \to B$.

Corollary 1.6. *Let A be a commutative k-algebra, $S \subseteq A$ a multiplicatively closed subset, and $f : A \to S^{-1}A$ the localization map. Assume that S does not contain any zero-divisors. Then $\mathrm{D}(f)$ induces an isomorphism*

$$\mathrm{D}(A) \cong \{P \in \mathrm{D}(S^{-1}A) : P * A \subseteq A\} \tag{1}$$

of filtered k-algebras.

Proof. See Másson [**23**], corollary 2.2.6. Notice that the proof also holds for any injective, formally étale homomorphism $f : A \to B$ of commutative k-algebras. $\square$

4. Quotient algebras

Let $f : A \to B$ be a surjective homomorphism of commutative k-algebras, and let $K = \ker(f)$ denote its kernel. Let furthermore $P \in \mathrm{End}_k(A)$ be a k-linear operator. Then P defines a k-linear operator $\overline{P} \in \mathrm{End}_k(B)$ given by the equation $\overline{P} * f(a) = f(P * a)$ for all $a \in A$ if and only if $P * K \subseteq K$. In this case, $\overline{P} = 0$ in $\mathrm{End}_k(B)$ if and only if $P * A \subseteq K$.

So let us assume that $P \in \mathrm{D}^p(A)$ is a differential operator such that $P * K \subseteq K$. Then $\overline{P} \in \mathrm{D}^p(B)$ is also a differential operator by induction on p. Consequently, there is an exact sequence of k-linear spaces

$$0 \to \mathrm{D}(A, K) \to \{P \in \mathrm{D}(A) : P * K \subseteq K\} \xrightarrow{f^*} \mathrm{D}(B),$$

where f^* is the homomorphism of filtered k-algebras given by $f^*(P) = \overline{P}$. When $\mathrm{D}(A)$ is better understood than $\mathrm{D}(B)$ and f^* is surjective, this gives a useful description of $\mathrm{D}(B)$. We shall see that this happens in many interesting cases.

We remark that $\mathrm{D}(A, K)$ is a right ideal in $\mathrm{D}(A)$, and there is an obvious inclusion of right ideals $K\,\mathrm{D}(A) \subseteq \mathrm{D}(A, K)$. If $A \otimes_k A$ is a Noetherian ring and $\mathcal{P}^p(A/k)$ is projective as a left A-module for all integers p, then $\mathrm{D}(A, K) = K\,\mathrm{D}(A)$, see Smith and Stafford [**32**], section 1.3 (e). We observe that if A is a free, commutative k-algebra, equality holds as well:

Lemma 1.7. *Let A be a free, commutative k-algebra. Then $\mathrm{D}(A, K) = K\,\mathrm{D}(A)$.*

Proof. We refer to the start of section 5 for the notation in this proof, and also for the description of $\mathrm{D}^p(A)$ (which is elementary). So let A be a free, commutative k-algebra on the symbols X, and let $P \in \mathrm{D}(A, K)$ be a differential operator of order $d(P) = p$. Then, $P \in F^p$, where F^p is as described in section 5, and we have that $P = \sum a_I \partial^I$, with $a_I \in A$ for all multi-indices I. We shall prove that $a_I \in K$ for all I: Assume that J is a multi-index such that $a_J \notin K$ and furthermore that $\deg(J)$ is minimal among the multi-indices with this property. Then we have that $P * x^J \in K$, and a calculation shows that

$$P * x^J = \sum_I a_I(\partial^I * x^J) = J!a_J + \sum_{I<J} J!/(J-I)!\, a_I x^{J-I}.$$

By the minimality of J, the last sum is in K and this is a contradiction. It follows that $P \in K\,\mathrm{D}(A)$. $\square$

We recall the notion of the *idealiser* of a right ideal J in an associative ring R: This is the ring $\mathbf{I}(J) = \{r \in R : rJ \subseteq J\} \subseteq R$, and it can be characterized as the largest subring of R containing J as an ideal.

Lemma 1.8. *Assume that* $\mathrm{D}(A, K) = K\,\mathrm{D}(A)$ *for an ideal* $K \subseteq A$. *Then the equality* $\mathbf{I}(K\,\mathrm{D}(A)) = \{P \in \mathrm{D}(A) : P * K \subseteq K\}$ *holds.*

PROOF. Assume that $P \in \mathbf{I}(K\,\mathrm{D}(A))$. Then $Px \in K\,\mathrm{D}(A)$ for all $x \in K$, since $K \subseteq K\,\mathrm{D}(A)$. But then $P*x = Px*1 \subseteq K\,\mathrm{D}(A)*1 \subseteq K$ for all $x \in K$, so $P*K \subseteq K$. Conversely, assume that $P * K \subseteq K$. Then, for all $x \in K$, $a \in A$, we have that $P * (xa) \in K$ from the assumption. But $P * (xa) = (Px) * a$, so $(Px) * A \subseteq K$ for all $x \in K$. Consequently, $Px \in \mathrm{D}(A, K) = K\,\mathrm{D}(A)$, and $PK\,\mathrm{D}(A) \subseteq K\,\mathrm{D}(A)$. $\square$

From lemma 1.8 and the preceding comment together with lemma 1.7, we see that if A is a free, commutative k-algebra, or A is such that $A \otimes_k A$ is Noetherian and $\mathcal{P}^p(A/k)$ is a projective A-module for all integers p, then there is a injective k-algebra homomorphism of filtered k-algebras $f^* : \mathbf{I}(K\,\mathrm{D}(A))/K\,\mathrm{D}(A) \to \mathrm{D}(B)$. The filtration of $\mathbf{I}(K\,\mathrm{D}(A))/K\,\mathrm{D}(A)$ is the natural one induced by the order filtration of $\mathrm{D}(A)$. We use this filtration to define the order of an equivalence class of differential operators in $\mathbf{I}(K\,\mathrm{D}(A))/K\,\mathrm{D}(A)$, and we remark that the order of such an equivalence class of differential operators equals the minimal order of the differential operators in the equivalence class.

Proposition 1.9. *Let* $f : A \to B$ *be a surjective homomorphism of commutative* k*-algebras. If* A *is a free, commutative* k*-algebra, or if* A *is such that* $A \otimes_k A$ *is Noetherian and* $\mathcal{P}^p(A/k)$ *is a projective* A*-module for all* $p \geq 0$, *then the* k*-algebra homomorphism* $f^* : \mathbf{I}(K\,\mathrm{D}(A))/K\,\mathrm{D}(A) \to \mathrm{D}(B)$ *is an isomorphism of filtered* k*-algebras.*

PROOF. We have to show that the k-algebra homomorphism f^* is surjective, and that the inverse is a homomorphism of filtered k-algebras. So we consider the exact sequence

$$0 \to K \to A \xrightarrow{f} B \to 0,$$

and apply the left exact functor $\mathrm{D}_A(-, B)$ to it, where B is considered as a left A-module. We obtain an exact sequence

$$0 \to \mathrm{D}_A(B, B) \to \mathrm{D}_A(A, B) \to \mathrm{D}_A(K, B).$$

Since f is surjective, we have that $\mathrm{D}_A(B, B) = \mathrm{D}_B(B, B) = \mathrm{D}(B)$. Let $P \in \mathrm{D}^p(B)$ be any differential operator of order at most p, and denote by Q its image in $\mathrm{D}_A(A, B)$ in the above exact sequence. Then $Q \in \mathrm{D}^p_A(A, B)$ since the co-domain of the functor is that of filtered modules, and $Q * K = 0$ from the exactness of the sequence. Notice that if A is free, then $\mathcal{P}^p(A/k)$ is a projective A-module for all integers p. This follows from Muhasky [**26**], lemma 1.4, or directly from the definition. So in all cases, $\mathcal{P}^p(A/k)$ is projective, and we see that there is a differential operator $Q' \in \mathrm{D}^p(A)$ such that $fQ' = Q$. Clearly, $Q' * K \subseteq K$ since $Q * K = 0$, so $P \mapsto Q'$ is the desired inverse of f^*. $\square$

Corollary 1.10. *Let* $f : A \to B$ *be a surjective homomorphism of commutative* k*-algebras. If* A *is a free, commutative* k*-algebra or if* A *is a regular* k*-algebra of finite type, then there is an isomorphism of filtered* k*-algebras*

$$f^* : \mathbf{I}(K\,\mathrm{D}(A))/K\,\mathrm{D}(A) \to \mathrm{D}(B). \tag{2}$$

PROOF. The first case is contained in proposition 1.9. For the second case, it is clear that $A \otimes_k A$ is Noetherian, so it only remains to check if $\mathcal{P}^p(A/k)$ is projective for all integers $p \geq 0$. Since A is of finite type over k, $\mathcal{P}^p(A/k)$ is an A-module of finite type by Másson [**23**], proposition 2.2.12. Then $\mathcal{P}^p(A/k)$ is projective if and only if it is locally free (or equivalently, locally projective). But from Másson [**23**], theorem 2.2.2, localizations commute with formation of principle parts. It is therefore enough to see that the p'th principle parts of A_m over k is a projective A_m-module for all maximal ideals $m \subseteq A$ and all integers p. But since A_m is a regular local ring, this follows from Smith and Stafford [**32**], section 1.3 (f). $\square$

5. Coordinatization

Let C be a free, commutative k-algebra on the symbols X, where X is any set. We shall give a description of C and the ring $\mathrm{D}(C)$ of differential operators on C explicitly in this case. We recall that if X is a finite set with n elements $x_1, \dots, x_n$, then C is the polynomial ring $C = k[x_1, \dots, x_n]$ and $\mathrm{D}(C)$ is the n'th Weyl algebra $\mathrm{D}(C) = k[x_1, \dots, x_n] < \partial_1, \dots, \partial_n >$, where we write ∂_i for $\partial/\partial x_i$. We recall that the relations in the n'th Weyl algebra is given by $[\partial_i, x_i] = 1$ for $1 \leq i \leq n$.

Let us write $X = \{x_\lambda\}_{\lambda \in \Lambda}$, where Λ is a set of indices for X. We define a multi-index I to be a function $I : \Lambda \to \mathbf{N_0}$, such that $I(\lambda) = 0$ for all except a finite number of elements $\lambda \in \Lambda$. Let x^I be the finite product

$$x^I = \prod_{\lambda \in \Lambda} x_\lambda^{I(\lambda)}$$

for all multi-indices I. Then, any element $c \in C$ can be written uniquely as a sum

$$c = \sum_I c_I x^I,$$

where $c_I \in k$ for all multi-indices I, and $c_I = 0$ for all except a finite number of indices. In particular, the set of monomials x^I for all multi-indices I is a basis for C, and the multiplication in C is the natural one.

Let ∂_λ be the k-linear operator $\partial_\lambda = \partial/\partial x_\lambda$ on C given by formal derivation with respect to x_λ. This is a well-defined k-linear derivation on C for all $\lambda \in \Lambda$, and we see that $\partial_\lambda, \partial_{\lambda'}$ commute for all $\lambda, \lambda' \in \Lambda$. Let ∂^I be the differential operator

$$\partial^I = \prod_{\lambda \in \Lambda} \partial_\lambda^{I(\lambda)}$$

for all multi-indices I. We shall write $\deg(I) = \sum I(\lambda)$ for the degree of a multi-index I, and this is a non-negative integer. Using this notation, it is clear that ∂^I is a differential operator of order at most $\deg(I)$, so $\partial^I \in \mathrm{D}^p(C)$ with $p = \deg(I)$. We shall also write $I! = \prod I(\lambda)!$ for all multi-indices I. With this convention, we see that the following formulas hold for all multi-indices I, J:

$$\partial^I * x^J = \begin{cases} J!/(J-I)!\, x^{J-I}, & I \leq J \\ 0, & \text{otherwise} \end{cases}$$

Notice that we write $I \leq J$ if $I(\lambda) \leq J(\lambda)$ for all $\lambda \in \Lambda$.

Let p be a fixed, non-negative integer, and let F^p be the k-vector space given by

$$F^p = \{\sum_I f_I \partial^I : f_I \in A,\ \deg(I) \leq p\}.$$

Notice that we allow infinite sums in this case. The reason for this is simple: Even if an element of F^p has an infinite number of non-zero terms, the evaluation of it on any given element $c \in C$ will give a finite sum. Hence, each element of F^p defines a k-linear operator on C.

Lemma 1.11. *We have $\mathrm{D}^p(C) = F^p$ for all integers $p \geq 0$.*

Proof. See Muhasky [**26**], Example 1.3. $\square$

We conclude that there is a multiplication $F^pF^q \subseteq F^{p+q}$. Clearly, we have that $x_\lambda, x_{\lambda'}$ commute and $\partial_\lambda, \partial_{\lambda'}$ commute for all $\lambda, \lambda' \in \Lambda$. Furthermore, $x_\lambda, \partial_{\lambda'}$ commute if $\lambda \neq \lambda'$, and

$$[\partial_\lambda, x_\lambda] = 1$$

for all $\lambda \in \Lambda$. We see that these formulas define the multiplication above. We also remark that lemma 1.11 implies that $d(\partial^I) = \deg(I)$, which will be useful later.

Let A be any commutative k-algebra. Then, a *set of coordinates* for A is a free, commutative k-algebra C together with a surjective k-algebra homomorphism $f : C \to A$. Clearly, a set of coordinates for A always exists, and for the rest of this section, we shall fix the coordinates $f : C \to A$. From corollary 1.10, we know that we can describe $\mathrm{D}(A)$ in the following way:

$$\mathrm{D}(A) \cong \mathbf{I}(K\,\mathrm{D}(C))/K\,\mathrm{D}(C),$$

where $K = \ker(f) \subseteq C$. Furthermore, the above identification is an isomorphism of filtered k-algebras. Let us write C as a free, commutative k-algebra on the symbols X, where $X = \{x_\lambda : \lambda \in \Lambda\}$. Then $Y = \{y_\lambda : \lambda \in \Lambda\}$ with $y_\lambda = f(x_\lambda)$ is a set of generators for A as algebra over k.

Assume that A is a commutative k-algebra of finite type. Then there is a set of coordinates $f : C \to A$ for A such that C is a polynomial ring over k in a finite number of variables $x_1, \dots, x_n$. Then $\mathrm{D}(C)$ is called the n'th Weyl algebra $A_n(k)$, and it is well-known that it is Noetherian and a finitely generated k-algebra. We remark that even if $\mathrm{D}(A)$ is a sub-quotient of $A_n(k)$ when A is of finite type over k, $\mathrm{D}(A)$ is not in general a finitely generated k-algebra. In fact, there is a famous counterexample given by Bernstein, Gelfand and Gelfand [**2**]: If A is the cubic cone $A = k[x, y, z]/(x^3 + y^3 + z^3)$, then $\mathrm{D}(A)$ is neither a finitely generated k-algebra nor a left or right Noetherian ring.

Let V be the k-vector space $V = \{\sum a_I \partial^I : a_I \in A \text{ for all multi-indices } I\}$, where we allow infinite sums. Then there is a unique, k-linear map $s_f : \mathrm{D}(A) \to V$ defined with respect to the fixed coordinates $f : C \to A$: For any differential operator $P \in \mathrm{D}(A)$, there exists a differential operator $P' \in \mathbf{I}(K\,\mathrm{D}(C))$ such that $f^*(P') = P$ by corollary 1.10. Let $p = d(P)$, then $P' \in F^p$ as defined above, so P' has the form

$$P' = \sum_I c_I \partial^I,$$

with $c_I \in C$ for all multi-indices I. Furthermore, P' is uniquely defined modulo $K\,\mathrm{D}(C)$, so the expression

$$s_f(P) = \sum_I f(c_I)\partial^I$$

defines a unique element in V. We say that $s_f(P)$ is the *standard form* of P with respect to the coordinates $f : C \to A$. It is straight-forward to check that s_f is an injective k-linear map.

Let $P \in \mathrm{D}(A)$ be any differential operator, and let $s_f(P) = \sum a_I \partial^I$ be the standard form of P with respect to the coordinates $f : C \to A$. Then we have that $d(P) = \max\{\deg(I) : a_I \neq 0\}$: First, notice that a differential operator $P' = \sum c_I \partial^I$ in $\mathrm{D}(C)$ has order $d(P') = \max\{\deg(I) : c_I \neq 0\}$. Secondly, we see that the minimal order among the differential operators in the equivalence class P' mod $K\,\mathrm{D}(C)$ is exactly $\max\{\deg(I) : a_I \neq 0\}$.

Let $\{\xi_\lambda\}$ be algebraically independent variables over A, and let $\xi^I = \prod \xi_\lambda^{I(\lambda)}$ for all multi-indices I. We define B to be the commutative A-algebra given by

$$B = \{\sum a_I \xi^I : a_I \in A,\ \max\{\deg(I) : a_I \neq 0\} < \infty\}.$$

Then, $B = \oplus B_p$ is a positively graded A-algebra, with graded components given by

$$B_p = \{ \sum_{\deg(I)=p} a_I \xi^I : a_I \in A\}$$

for all $p \geq 0$. In particular, $B_0 = A$.

For all differential operators $P \in \mathrm{D}^p(A)$ for $p \geq 0$, let σ_p be the homogeneous form of degree p in B defined by

$$\sigma_p(P) = \sum_{\deg(I)=p} a_I \xi^I$$

where the coefficients a_I are given by the standard form $s_f(P) = \sum a_I \partial^I$ of P. We define the *symbol* of a non-zero differential operator $P \in \mathrm{D}(A)$ with respect to the coordinates $f : C \to A$ to be $\sigma(P) = \sigma_p(P)$, where $p = d(P)$. If $Q \in \mathrm{D}(A)$ is another differential operator, with $d(Q) < d(P)$, we see that $\sigma(P+Q) = \sigma(P)$. Furthermore, we have that $\sigma_{p+q}(PQ) = \sigma_p(P)\sigma_q(Q)$ for all non-zero differential operators $P, Q \in \mathrm{D}(A)$ with $p = d(P)$, $q = d(Q)$. Consequently, σ defines an injective homomorphism of graded k-algebras

$$\sigma : \operatorname{gr} \mathrm{D}(A) \to B,$$

given by $\overline{P} \mapsto \sigma(P)$ for all non-zero differential operators $P \in \operatorname{gr} \mathrm{D}(A)$.

Proposition 1.12. *Let A be a commutative k-algebra. If A is essentially of finite type over k, then $\operatorname{gr} \mathrm{D}(A)$ is an integral domain if and only if A is an integral domain.*

Proof. We have inclusions of k-algebras $A \subseteq \operatorname{gr} \mathrm{D}(A) \subseteq B$. Assume that A is of finite type over k. Then we may choose the coordinates $f : C \to A$ such that Λ is a finite set, which means that B is an integral domain if A is an integral domain: This is well-known, since B is a polynomial ring over A in a finite number of variables. On the other hand, we know that $\mathrm{D}(S^{-1}A) \cong S^{-1}A \otimes_A \mathrm{D}(A)$, so $\operatorname{gr} \mathrm{D}(S^{-1}A) \cong S^{-1} \operatorname{gr} \mathrm{D}(A)$. Consequently, if $\operatorname{gr} \mathrm{D}(A)$ is an integral domain, so is $S^{-1} \operatorname{gr} \mathrm{D}(A)$, and the rest is clear. □

Finally, we shall refine the description of the order of differential operators given in lemma 1.1. By using the coordinates $f : C \to A$, we obtain the following result:

Proposition 1.13. *Let A be a commutative k-algebra, and $P, Q \in \mathrm{D}(A)$ be differential operators. Then we have:*
i) $d(P+Q) \leq \max\{d(P), d(Q)\}$,
ii) $d(P+Q) < \max\{d(P), d(Q)\}$ *if and only if* $d(P) = d(Q)$ *and* $\sigma(P) = -\sigma(Q)$,
iii) $d(PQ) \leq d(P) + d(Q)$,
iv) $d([P,Q]) \leq d(P) + d(Q) - 1$,
Furthermore, if A is an integral domain, essentially of finite type over k, then we have $d(PQ) = d(P) + d(Q)$.

Proof. Claims i), iii) and iv) are contained in lemma 1.1. The second claim is easily obtained by writing P, Q in standard form with respect to the coordinates $f : C \to A$, and using the formula for order in terms of coordinates. The last part is simply a reformulation of proposition 1.12, since $d(PQ) = d(P) + d(Q)$ holds for all $P, Q \in \mathrm{D}(A)$ if and only $\operatorname{gr} \mathrm{D}(A)$ is an integral domain. □

6. Graded algebras

Let A be a $\mathbf{Z}$-graded k-algebra $A = \oplus A_i$. Then A_0 is a k-algebra, and A_i is an A_0-module for all integers i. We say that A is *positively graded* if $A_i = 0$ for all integers $i < 0$, and that A is *quasi-homogeneous* if A is positively graded and $A_0 = k$.

Let $P \in \mathrm{D}(A)$ be a differential operator. We say that P is *homogeneous of weight* w for some integer w if $P * A_i \subseteq A_{i+w}$ for all integers i, and we shall write $\mathrm{D}(A)_w$ for the set of all homogeneous differential operators of weight w. Then $\mathrm{D}(A)_w \subseteq \mathrm{D}(A)$ is a k-linear subspace. We shall also write $\mathrm{D}^p(A)_w = \mathrm{D}(A)_w \cap \mathrm{D}^p(A)$ for the k-linear subspace of differential operator of weight w and order at most p for all integers p, w. We remark that $\mathrm{D}(A)_w \, \mathrm{D}(A)_{w'} \subseteq \mathrm{D}(A)_{w+w'}$ for all integers w, w', and similarly we have $\mathrm{D}^p(A)_w \, \mathrm{D}^q(A)_{w'} \subseteq \mathrm{D}^{p+q}(A)_{w+w'}$ for all integers p, q, w, w'.

Let C be the free, commutative k-algebra on the symbols X, where X is the set $X = \{x_\lambda : \lambda \in \Lambda\}$ in the notation from section 5. We shall consider this as a $\mathbf{Z}$-graded k-algebra $C = \oplus C_i$ by assigning an integer weight w_λ to each free variable x_λ in C. Let us denote by $w(I)$ the weighted sum $w(I) = \sum w_\lambda I(\lambda)$ for each multi-index I. Then $w(I)$ is an integer, and each x^I is an homogeneous element of C of weight $i = w(I)$. In particular, we see that the set of monomials x^I such that $w(I) = i$ is a basis for C_i.

For all $\lambda \in \Lambda$, let e_λ denote the multi-index defined by $e_\lambda(\lambda) = 1$ and $e_\lambda(\lambda') = 0$ for all $\lambda' \neq \lambda$. Then, we have that $\partial_\lambda * x^I = I(\lambda) x^{I - e_\lambda}$ for all multi-indices I with $I(\lambda) \geq 1$, and zero otherwise. It follows that ∂_λ is a homogeneous differential operator in $\mathrm{D}(C)$ of weight $-w_\lambda$. Consequently, any differential operator $x^I \partial^{I'}$ in $\mathrm{D}(C)$ is homogeneous of weight $w(I) - w(I')$, and we have

$$\mathrm{D}^p(C)_w = \{ \sum_{\deg(I) \leq p} c_I \partial^I : c_I \in C_{w + w(I)} \}$$

for all integers p, w.

Proposition 1.14. *Let $f : A \to B$ be a homomorphism of commutative, $\mathbf{Z}$-graded k-algebras which is homogeneous of degree 0. Then we have:*

i) If f is an injective, formally étale homomorphism, then the identification (1) of k-algebras $\mathrm{D}(f) : \mathrm{D}(A) \to \{P \in \mathrm{D}(B) : P * A \subseteq A\}$ from corollary 1.6 preserves homogeneous elements and their weights.

ii) If A is a free, commutative k-algebra or a regular k-algebra of finite type over k, and if f is a surjective homomorphism with kernel K, then the identification (2) of k-algebras $f^* : \mathbf{I}(K \, \mathrm{D}(A))/K \, \mathrm{D}(A) \to \mathrm{D}(B)$ from corollary 1.10 preserves homogeneous elements and their weights.

In particular, the identification (1) exists and preserves homogeneous elements and their weights if $B = S^{-1}A$ for a multiplicatively closed subset $S \subseteq A$ which does not contain any zero-divisors and f is the canonical localization map.

Proof. In the first case, a differential operator in $\{P \in \mathrm{D}(B) : P * A \subseteq A\}$ is homogeneous of weight w if it is homogeneous of weight w as a differential operator in $\mathrm{D}(B)$. Similarly, in the second case, an equivalence class of differential operators in $\mathbf{I}(K \, \mathrm{D}(A))/K \, \mathrm{D}(A)$ is homogeneous of weight w if each homogeneous differential operator in the equivalence class is homogeneous of weight w as a differential operator in $\mathrm{D}(A)$. With this observation in mind, it is enough to see that in each case, the extended differential operator $\overline{P}$ is characterized by the equation $\overline{P} * f(a) = f(P * a)$ for all $a \in A$. □

Let A be a commutative k-algebra. A *homogeneous set of coordinates* for A is a free, commutative, $\mathbf{Z}$-graded k-algebra C, together with a k-algebra homomorphism $f : C \to A$ which is homogeneous of degree 0 for the chosen weights w_λ. So any

Z-graded k-algebra A admits a set of homogeneous coordinates, and for the rest of this section, we shall fix the homogeneous coordinates $f : C \to A$ and the associated weights w_λ.

Let $P \in \mathrm{D}(A)$ be a non-zero differential operator. From proposition 1.14, we see that P is homogeneous of weight w if and only if the standard form of P with respect to the coordinates $f : C \to A$ is of the form

$$s(P) = \sum a_I \partial^I,$$

where $a_I \in A_{w+w(I)}$ for all multi-indices I. Furthermore, we see that any non-zero differential operator $P \in \mathrm{D}(A)$ has the form $P = \sum P_w$ with each P_w homogeneous of weight w, and if Λ is a finite set, we may assume that this sum is finite.

Lemma 1.15. *Assume that A is of finite type over k. For all integers p, we have*
i) $\mathrm{D}(A) = \oplus \mathrm{D}(A)_w$,
ii) $\mathrm{D}^p(A) = \oplus \mathrm{D}^p(A)_w$,
where the two direct sums run over all integers w.

Proof. Since A is of finite type over k, there is a finite set of homogeneous coordinates for A, that is, we may assume that Λ is finite. The rest is clear, since $\mathrm{D}(A)_w \cap \mathrm{D}(A)_{w'} = 0$ if $w \neq w'$. □

For the rest of this section, we shall assume that A is of finite type over k. Then, $\mathrm{D}(A)$ is a graded k-algebra from lemma 1.15, and $\mathrm{D}^p(A)$ is a graded A-module. We also obtain the following result:

Corollary 1.16. *Let $f : A \to B$ be a homomorphism of commutative, **Z**-graded k-algebras of finite type. If f is injective and formally étale, then the identification (1) from corollary 1.6 is an isomorphism of **Z**-graded k-algebras. If A is a free, commutative k-algebra or a regular k-algebra, and if f is surjective, then the identification (2) from corollary 1.10 is an isomorphism of **Z**-graded k-algebras.*

Proof. In the case of a injective, formally étale homomorphism, it is clear that $\{P \in \mathrm{D}(B) : P * A \subseteq A\}$ is a graded subring of $\mathrm{D}(B)$. In the case of a surjective homomorphism, it is clear that $\mathbf{I}(K\,\mathrm{D}(A))/K\,\mathrm{D}(A)$ is a graded subquotient of $\mathrm{D}(A)$. So all k-algebras involved have natural **Z**-gradings. The rest is clear. □

From lemma 1.15, it also follows that the associated graded ring $\mathrm{gr}\,\mathrm{D}(A)$ has a natural $\mathbf{Z}^2$-grading, since we have

$$\mathrm{gr}\,\mathrm{D}(A) = \bigoplus_{p,w} \mathrm{D}^p(A)_w / \mathrm{D}^{p-1}(A)_w.$$

We say that $x \in \mathrm{gr}\,\mathrm{D}(A)$ is homogeneous of bidegree (p, w) if $x \in \mathrm{D}^p(A)_w / \mathrm{D}^{p-1}(A)_w$ for some integers $p, w \in \mathbf{Z}$. Furthermore, we denote by $\mathrm{gr}^{(p,w)}\,\mathrm{D}(A)$ the linear subspace in $\mathrm{gr}\,\mathrm{D}(A)$ of homogeneous elements of bidegree (p, w). We also notice that if $C = k[x_1, \dots, x_n]$, where x_i is homogeneous of degree w_i, then $B = A[\xi_1, \dots, \xi_n]$, and we consider this as a $\mathbf{Z}^2$-graded ring: Every homogeneous element in A of weight w is homogeneous in B of bidegree $(0, w)$, and ξ_i is homogeneous in B of bidegree $(1, -w_i)$. It follows that the symbol relative to the homogeneous coordinates f defines an injective homomorphism $\sigma : \mathrm{gr}\,\mathrm{D}(A) \to A[\xi_1, \dots, \xi_n]$ of $\mathbf{Z}^2$-graded k-algebras.

Let $f : A \to B$ be an injective, formally étale homomorphism of **Z**-graded k-algebras. Then $\mathrm{D}(f) : \mathrm{D}(A) \to \mathrm{D}(B)$ is a injective homomorphism of **Z**-graded k-algebras such that $d(\mathrm{D}(f)(P)) = d(P)$ for all differential operators $P \in \mathrm{D}(A)$ by corollary 1.6 and corollary 1.16. Consequently, this homomorphism induces an injection of $\mathbf{Z}^2$-graded k-algebras

$$\mathrm{gr}(f) : \mathrm{gr}\,\mathrm{D}(A) \to \mathrm{gr}\,\mathrm{D}(B)$$

given by $\operatorname{gr}(f)(\overline{P}) = \overline{\mathrm{D}(f)(P)}$ for all non-zero differential operators $P \in \mathrm{D}(A)$. We shall therefore consider $\operatorname{gr}\mathrm{D}(A)$ a $\mathbf{Z}^2$-graded subring of $\operatorname{gr}\mathrm{D}(B)$ whenever $f : A \to B$ is an injective, formally étale homomorphism of $\mathbf{Z}$-graded k-algebras. In particular, this applies when A is a $\mathbf{Z}$-graded k-algebra, $B = S^{-1}A$ is the localization with respect to a multiplicatively closed subset $S \subseteq A$ consisting of homogeneous elements, and $f : A \to B$ is the localization map.

We end this chapter with the definition of the *Bernstein filtration*, an alternative to the usual order filtration of $\mathrm{D}(A)$ in the graded case, with nice computational advantages. Assume that A is a $\mathbf{Z}$-graded k-algebra of finite type. Then the n'th k-linear subspace $\mathrm{B}^n(A) \subseteq \mathrm{D}(A)$ in the Bernstein filtration is given by

$$\mathrm{B}^n(A) = \bigoplus_{w \in \mathbf{Z}} \mathrm{D}^{(n-w)/2}(A)_w,$$

where $\mathrm{D}^q(A)_w = \{P \in \mathrm{D}(A)_w : d(P) \leq q\}$ for all rational numbers q. The motivation for this definition is the following special case: If $A = k[t]$ with the usual $\mathbf{Z}$-grading, then the n'th filtered subspace in the Bernstein filtration of $\mathrm{D}(k[t])$ has k-linear basis $\{t^a \partial^b : a + b \leq n\}$.

We see that the linear subspaces $\mathrm{B}^n(A)$ form an ascending filtration of the k-algebra $\mathrm{D}(A)$. We may therefore consider the associated graded ring associated with this filtration,

$$\operatorname{gr}' \mathrm{D}(A) = \bigoplus_n \mathrm{B}^n(A)/\,\mathrm{B}^{n-1}(A),$$

which is a $\mathbf{Z}$-graded k-algebra. Furthermore, we see that there is an isomorphism of k-algebras between $\operatorname{gr}' \mathrm{D}(A)$ and the usual associated graded ring $\operatorname{gr}\mathrm{D}(A)$, and a homogeneous element in $\operatorname{gr}\mathrm{D}(A)$ of bidegree (p, w) corresponds to a homogeneous element in $\operatorname{gr}'\mathrm{D}(A)$ of degree $n = 2p + w$.

Proposition 1.17. *Let A be a quasi-homogeneous k-algebra of finite type over k. Then $\mathrm{D}^p(A)_w$ is a k-vector space of finite dimension for all integers p, w. In particular, $\operatorname{gr}^{(p,w)} \mathrm{D}(A)$ is a k-vector space of finite dimension.*

Proof. Notice that A is quasi-homogeneous if and only if the weights $w_\lambda > 0$ for all $\lambda \in \Lambda$ and for all choices of homogeneous coordinates. Consequently, the set $\{I : w(I) = c\}$ is finite for any given integer c. But the set

$$\{x^I \partial^J : \deg(J) \leq p,\ w(I) - w(J) = w\}$$

is a basis for $\mathrm{D}^p(C)_w$, and this basis is finite: The number of multi-indices J such that $\deg(J) \leq p$ is finite, and for each such J, there is only a finite number of multi-indices I such that $w(I) = w + w(J)$. Consequently, there is a finite basis for $\mathrm{D}^p(A)_w$ as well. The rest is clear. □

A very useful observation is the following: If A is quasi-homogeneous, and if $\mathrm{D}^{(n-w)/2}(A)_w$ is non-zero for a finite number of weights w, then $\mathrm{B}^n(A)$ has finite dimension as well. In the next chapter, we will show that in some interesting examples, this condition is fulfilled for all integers n.

CHAPTER 2

Differential operators on monomial curves

In this chapter, we give an explicit description of the ring $\mathrm{D}(A)$ of differential operators on A, its associated graded ring $\operatorname{gr}\mathrm{D}(A)$ and the module of derivations $\operatorname{Der}_k(A)$ when $A = k[\Gamma]$ is the affine coordinate ring of a monomial curve. We find methods and obtain results that make it possible to do explicit calculations with differential operators on monomial curves. The results described in this chapter were obtained in a joint work with Henrik Vosegaard. We are aware that some special rings of differential operators on monomial curves had been calculated by I. Musson and others, and that some of the results given in this chapter can also be found in Jones [**17**].

There are some strong structural results on the ring of differential operators of affine algebraic curves, which appeared in Smith and Stafford [**32**]. In particular, the ring of differential operators on a monomial curve is Morita equivalent to the first Weyl algebra $\mathrm{A}_1(k)$. We mention some of these results in this chapter, for later reference.

1. Monomial curves

Let Γ be a finitely generated semigroup with an embedding $\Gamma \subseteq \mathbf{Z}^m$. Then clearly, $\mathbf{Z}\Gamma \cong \mathbf{Z}^n$ for some non-negative integer $n \leq m$. So by changing the embedding, we may assume that $\mathbf{Z}\Gamma = \mathbf{Z}^n$. We denote by an *affine semigroup* a finitely generated semigroup with fixed embedding $\Gamma \subseteq \mathbf{Z}^n$ such that $\mathbf{Z}\Gamma = \mathbf{Z}^n$. By convention, all semigroups are assumed to have an additive identity 0.

Let $\Gamma \subseteq \mathbf{Z}^n$ be an affine semigroup. The *semigroup algebra* $A = k[\Gamma]$ is defined to be the k-algebra in $T = k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ with k-linear basis $\{t^\gamma : \gamma \in \Gamma\}$, where t^γ is given by

$$t^\gamma = \prod_{i=1}^{n} t_i^{\gamma_i}$$

with $\gamma = (\gamma_1, \dots, \gamma_n)$ as an element of $\mathbf{Z}^n$. The multiplication in A, inherited from the multiplication in T, is given by $t^\gamma t^{\gamma'} = t^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. In particular, we see that $k[\mathbf{Z}^n] = T$.

Since Γ is an commutative semigroup of finite type, we see that $A = k[\Gamma]$ is a commutative k-algebra of finite type. Since A is a subring of the integral domain T, we also see that A is an integral domain. In particular, it follows that $A = k[\Gamma]$ is the affine coordinate ring of an affine algebraic variety $X_\Gamma = \operatorname{Spec}(A)$. This is the origin of the name affine semigroup. We remark that the Krull dimension of A, and hence the dimension of X_Γ, is n: This is easy to see, since the integral domains A, T and $k[t_1, \dots, t_n]$ have the same field of fractions, and hence the same Krull dimension.

The k-algebra T has a natural $\mathbf{Z}$-grading such that t_i is homogeneous of degree 1 and t_i^{-1} is homogeneous of degree -1 for $1 \leq i \leq n$. From the construction of $A = k[\Gamma]$, we see that $A \subseteq T$ has a natural structure as a graded subring: Let G be a finite set of generators for Γ. Then $\{t^\gamma : \gamma \in G\}$ is a set of generators of A as a

k-algebra, and each t^γ is homogeneous in T: For all $\gamma \in \Gamma$, write $|\gamma| = \sum \gamma_i$, where $\gamma = (\gamma_1, \dots, \gamma_n)$ as an element in $\mathbf{Z}^n$. Then t^γ is homogeneous in T of degree $|\gamma|$.

We also see that there exists a multiplicatively closed subset $S \subseteq A$ consisting of homogeneous elements of A, such that $T \cong S^{-1}A$: This is clear, since $\mathbf{Z}\Gamma = \mathbf{Z}^n$. In fact, we may choose S to be the multiplicatively closed subset of A generated by a set of homogeneous generators for A as k-algebra.

Let us consider the case special case $n = 1$: That is, let $\Gamma \subseteq \mathbf{Z}$ be an affine semigroup. Then, we have the following structural result:

Lemma 2.1. *Let $\Gamma \subseteq \mathbf{Z}$ be an affine semigroup. Then Γ is a semigroup of one of the following forms:*
i) $\Gamma \subseteq \mathbf{N_0}$*, and* $n + \mathbf{N_0} \subseteq \Gamma$ *for some non-negative integer* n*,*
ii) $-\Gamma \subseteq \mathbf{N_0}$*, and* $n + \mathbf{N_0} \subseteq -\Gamma$ *for some non-negative integer* n*,*
iii) $\Gamma = \mathbf{Z}$*.*

Proof. Let $\Gamma \subseteq \mathbf{Z}$ be an affine semigroup. If $\Gamma \subseteq \mathbf{N_0}$, then it is easy to see that the condition $\mathbf{Z}\Gamma = \mathbf{Z}$ implies that $n + \mathbf{N_0} \subseteq \Gamma$ for some non-negative integer n (in fact, these conditions are equivalent). A similar statement holds if $-\Gamma \subseteq \mathbf{N_0}$. So it is enough to prove that if Γ contains strictly positive and strictly negative elements, then $\Gamma = \mathbf{Z}$: Assume that Γ contains strictly positive and strictly negative elements, and consider the semigroup $\Gamma' = \Gamma \cap \mathbf{N_0} \subseteq \mathbf{N_0}$. Then, there exists a unique positive integer d, such that $\Gamma' \subseteq d\mathbf{N_0}$ and $d(n + \mathbf{N_0}) \subseteq \Gamma'$ for some non-negative integer n. If $d = 1$, then $\Gamma = \mathbf{Z}$ and we are done. So assume that $d > 1$. For all positive integers m such that $-m \in \Gamma$, we have that $Nd - m \in \Gamma'$ for some positive integer N. Hence $\Gamma \subseteq d\mathbf{Z}$, so $\mathbf{Z}\Gamma \subseteq d\mathbf{Z}$, which is a contradiction. □

We say that an affine semigroup $\Gamma \subseteq \mathbf{Z}$ is a *numerical semigroup* if $\Gamma \subseteq \mathbf{N_0}$. From lemma 2.1, we see that for any numerical semigroup Γ, there exists a non-negative number n such that $n + \mathbf{N_0} \subseteq \Gamma$. Equivalently, if $\{g_1, \dots, g_m\}$ is a set of generators of Γ, then $(g_1, \dots, g_m) = 1$. A very useful characterization of numerical semigroups is the following: Any subsemigroup $\Gamma \subseteq \mathbf{N_0}$ such that $\mathbf{N_0} \setminus \Gamma$ is finite, is a numerical semigroup.

Let us recall some well-known facts about numerical semigroups, and at the same time fix the notations which we will use throughout this thesis: We shall denote by $c = c(\Gamma)$ the *conductor* of Γ, which is the least integer n such that $n + \mathbf{N_0} \subseteq \Gamma$. We shall denote by H the set of *holes* in Γ, defined by $H = \mathbf{N_0} \setminus \Gamma$. This is a finite, possibly empty set, and we denote by h its cardinality. If $\Gamma \neq \mathbf{N_0}$, we denote by $g = g(\Gamma)$ the *Frobenius number* of Γ, which is given by $g = \max H$. Clearly, we have that $g + 1 = c$.

A numerical semigroup Γ has a unique, minimal set of generators in a very strong sense: In fact, every set of generators for Γ contains this minimal set. We shall always denote the minimal set of generators by $\{a_1, \dots, a_r\}$, and we choose the order of the generators such that $a_1 < a_2 < \cdots < a_r$. Let us construct the set $\{a_1, \dots, a_r\}$ explicitly: We define a_1 to be

$$a_1 = \min\{\gamma \in \Gamma : \gamma \neq 0\},$$

and define a_{i+1} recursively as

$$a_{i+1} = \min\{\gamma \in \Gamma : \gamma \notin < a_1, \dots, a_i >\},$$

for all integers i such that $< a_1, \dots, a_i > \neq \Gamma$, where $< a_1, \dots, a_i >$ denotes the subsemigroup in Γ generated by $\{a_1, \dots, a_i\}$. Clearly, there exists some integer r such that $< a_1, \dots, a_r >= \Gamma$, so $\{a_1, \dots a_r\}$ is a generating set for Γ. From the construction, we see that the generators satisfy $a_1 < a_2 < \cdots < a_r$, and that they satisfy the minimality condition. The minimal number r of generators for Γ is called the *rank* of Γ.

We say that a numerical semigroup Γ is *symmetric* if there is an integer $n \in \mathbf{Z}$ such that $\gamma \in \Gamma$ if and only if $n - \gamma \notin \Gamma$ for all $\gamma \in \mathbf{Z}$. It is easy to see that if Γ is symmetric, then this integer n must equal the Frobenius number g.

Let Γ be a numerical semigroup, and consider the semigroup algebra $A = k[\Gamma]$. Since $n = 1$ in this case, we shall drop the index and write t for t_1. We see that $A = k[t^{a_1}, \dots, t^{a_r}]$ considered as a subring of $k[t]$. Furthermore, $X_\Gamma = \operatorname{Spec} A$ is a curve since $n = 1$, and in fact X_Γ is the parametrized curve

$$X_\Gamma = \{(x_1, \dots, x_r) \in \mathbf{A}^r : x_i = t^{a_i} \text{ with } t \in k \text{ for } 1 \leq i \leq r\},$$

of embedding dimension r. We say that an affine, algebraic variety is a *monomial curve* if it is isomorphic to X_Γ for some numerical semigroup Γ. By abuse of language, we shall also say that the affine coordinate ring $A = k[\Gamma]$ of X_Γ is a monomial curve.

Clearly, $k[t]$ is the integral closure of $A = k[\Gamma]$. We shall write $\overline{A} = k[t]$ for all numerical semigroups Γ, and refer to $\overline{A}$ as the *normalization* of A. Furthermore, we say that $\overline{X_\Gamma} = \operatorname{Spec} \overline{A}$ is the *normalization* of X_Γ, and that the morphism $\overline{X_\Gamma} \to X_\Gamma$ of affine varieties induced by the inclusion $A \subseteq \overline{A}$ is the *normalization map* of X_Γ. We see that $\overline{X_\Gamma} = \mathbf{A}^1$, and that the normalization morphism is given by $t \mapsto (t^{a_1}, \dots, t^{a_r})$ for all $t \in k$. It is easy to see that the normalization map of any monomial curve X_Γ is a bijection. But if $\Gamma \neq \mathbf{N_0}$, it is not an isomorphism of algebraic varieties.

As in the general case, $A = k[\Gamma]$ is a graded subring of $T = k[t, t^{-1}]$ for all numerical semigroups Γ. Furthermore, there is a multiplicatively closed subset S of A consisting of homogeneous elements, such that $S^{-1}A \cong T$: In fact, we may choose $S = \{x^n : n \geq 0\}$ with $x = t^{a_1}$. Clearly, all monomial curves $A = k[\Gamma]$ are quasi-homogeneous. We also see that A is regular if $\Gamma = \mathbf{N_0}$, and that A has an isolated singularity in 0 otherwise.

2. Differential operators

Let $\Gamma \subseteq \mathbf{Z}^n$ be an affine semigroup. Then $A = k[\Gamma] \subseteq T = k[t_1, t^{-1}, \dots, t_n, t_n^{-1}]$ is an inclusion of graded k-algebras, and $T = S^{-1}A$ is a localization of A with respect to a multiplicatively closed subset S consisting of homogeneous elements. So by the identification (1) in corollary 1.6, we see that

$$\mathrm{D}(A) = \{P \in \mathrm{D}(T) : P * A \subseteq A\}.$$

Moreover, we can use the same technique to describe $\mathrm{D}(T)$ explicitly in terms of $\mathrm{D}(B)$, where $B = k[\mathbf{N_0}^n] = k[t_1, \dots, t_n]$: Also in this case, we see that $B \subseteq T$ is a graded subalgebra, and that T is a localization $T = S^{-1}B$ with respect to some multiplicatively closed subset $S \subseteq B$ consisting of homogeneous elements in B. So from proposition 1.5, we see that there is an isomorphism of $\mathrm{D}(B)$-T bimodules

$$\mathrm{D}(T) \cong T \otimes_B \mathrm{D}(B).$$

But $\mathrm{D}(B)$ is the n'th Weyl algebra $\mathrm{D}(B) = A_n(k)$, so we have an explicit description of $\mathrm{D}(T)$: For all integers $p \geq 0$, any differential operator $P \in \mathrm{D}^p(T)$ can be written uniquely as a sum

$$P = \sum_{\deg(I) \leq p} c_I \partial^I$$

with $c_I \in T$ for all multi-indices I. We also remark that the inclusion $\mathrm{D}(A) \subseteq \mathrm{D}(T)$ is an inclusion of $\mathbf{Z}$-graded k-algebras from corollary 1.16.

Let $E_i = t_i \partial_i$, then E_i is a homogeneous derivation in $\mathrm{D}(T)$ of weight 0 for $1 \leq i \leq n$. We also see that $E_i * A \subseteq A$, since $E_i * t^I = I(i) t^I$ for all multi-indices I. Furthermore, all these derivations commute. Let us denote the k-algebra generated by $E_1, \dots, E_n$ in $\mathrm{D}(T)$ by W. Then W is isomorphic to a commutative polynomial ring in n variables over k, and clearly, $W \subseteq \mathrm{D}(A)_0$.

Let $w \in \mathbf{Z}^n$. We shall write $\Omega(w)$ for the subset $\Omega(w) \subseteq \Gamma$ given in the following way:

$$\Omega(w) = \{\gamma \in \Gamma : \gamma + w \notin \Gamma\}.$$

The semigroup Γ is considered as a subsemigroup of $\mathbf{Z}^n$, and the addition used in the definition of $\Omega(w)$ takes place in $\mathbf{Z}^n$. The set $\Omega(w)$ does of course not only depend on w, but also upon the affine semigroup Γ, even if this is suppressed from the notation.

We remark that we may view $\Omega(w)$ as a subset of $\mathbf{A}^n_k$: This is clear, since $\Gamma \subseteq \mathbf{Z}^n \subseteq \mathbf{A}^n_k$. Let us denote by $I(\Omega(w))$ the ideal defined by $\Omega(w)$ in the sense of classical algebraic geometry: It is an ideal in the polynomial ring $k[x_1, \dots, x_n]$, where $x_1, \dots, x_n$ are the coordinate functions on $\mathbf{A}^n_k$, and it consists of the functions in $k[x_1, \dots, x_n]$ which vanish on $\Omega(w)$. We shall identify this ideal with an ideal in W, by identifying the i'th coordinate function x_i with E_i for $1 \leq i \leq n$. By abuse of notation, we shall denote the new ideal in W by $I(\Omega(w))$ as well.

Theorem 2.2. *Let $\Gamma \subseteq \mathbf{Z}^n$ be an affine semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. Then we have*

$$\mathrm{D}(A) = \bigoplus_{w \in \mathbf{Z}^n} t^w I(\Omega(w)).$$

In particular, the k-linear subspace of $\mathrm{D}(A)$ consisting of homogeneous differential operators of weight m is given by

$$\mathrm{D}(A)_m = \bigoplus_{|w|=m} t^w I(\Omega(w))$$

for all integers $m \in \mathbf{Z}$.

Proof. See Musson, [**27**], theorem 2.3. □

We remark that we could consider $\mathrm{D}(A)$ a $\mathbf{Z}^n$-graded k-algebra, using the $\mathbf{Z}^n$-grading of A. Since we will mostly be interested in monomial curves, we have chosen not to develope the theory of diffential operators in that direction.

Let us consider the special case when $\Gamma \subseteq \mathbf{Z}$ is a numerical semigroup: In this case, we write ∂ for ∂_1 and $E = t\partial$ for the homogeneous derivation $E_1 = t_1\partial_1$ of weight 0 in $\mathrm{D}(A)$. We know that H is finite, so it is clear that $\Omega(w)$ is a finite set for all integers $w \in \mathbf{Z}$, and we shall write $\tau(w)$ for the cardinality of $\Omega(w)$ for all $w \in \mathbf{Z}$. This defines a numerical function $\tau : \mathbf{Z} \to \mathbf{N_0}$, which depends upon the numerical semigroup Γ. Notice that if $w \notin \Gamma$, then $0 \in \Omega(w)$. Consequently, we have that $\tau(w) = 0$ if and only if $w \in \Gamma$, so τ describes Γ completely.

For all integers w, we define the *characteristic polynomial* relative to w to be the polynomial $\chi_w \in k[\xi]$ given by

$$\chi_w(\xi) = \prod_{\gamma \in \Omega(w)} (\xi - \gamma).$$

This is a monic polynomial in $k[\xi]$ of degree $\tau(w)$, and it will of course depend upon the numerical semigroup Γ. Using this polynomial, we obtain the following special case of theorem 2.2:

Corollary 2.3. *Let Γ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. Then we have*

$$\mathrm{D}(A) = \bigoplus_{w \in \mathbf{Z}} P_w k[E],$$

where $P_w = t^w \chi_w(E)$. In particular, the k-linear subspace of $\mathrm{D}(A)$ consisting of homogeneous differential operators of weight w is given by $\mathrm{D}(A)_w = P_w k[E]$ for all integers w.

Proof. We have to prove that the ideal $I(\Omega(w)) \subseteq k[E]$ is generated by $\chi_w(E)$. But this is clear, since any polynomial in one variable that vanishes on $\Omega(w)$ must be a multiple of χ_w. □

3. The numerical function τ

For the rest of this chapter, we shall assume that Γ is a numerical semigroup, and that $A = k[\Gamma]$ is the corresponding monomial curve. Notice that since A is an integral domain of finite type, $E^n \in \mathrm{D}(A)$ has order $d(E^n) = n$ for all positive integers n by proposition 1.13. So in particular, P_w is a differential operator in $\mathrm{D}(A)$ of order $d(P_w) = \tau(w)$ for all integers w by corollary 2.3. Since $\tau(w)$ is the minimal order of a homogeneous differential operator in $\mathrm{D}(A)$ of weight w, it is clearly of interest to study the behaviour of the numerical function τ.

Proposition 2.4. *For all integers w, we have $\tau(w) + w = \tau(-w)$.*

Proof. Since the formula is symmetric, and obviously true for $w = 0$, we may assume that $w > 0$. For $0 \leq j \leq w - 1$, we define $m_j = \min\ \Gamma \cap (j + w\mathbf{N_0})$. It follows that $m_0, \dots, m_{w-1}$ are distinct integers in $\Omega(-w)$. It is therefore enough to construct a bijection $f : \Omega(-w) \setminus \{m_0, \dots, m_{w-1}\} \to \Omega(w)$. So let $l \in \Omega(-w)$ be different from all the m_i's. Then $l \in \Gamma$, $l - w \notin \Gamma$, but there exists some integer $n > 1$ such that $l - nw \in \Gamma$. Let n_l be the least of all such positive numbers n, and put $f(l) = l - n_l w$. Then $f(l) \in \Omega(w)$, and f is a well-defined map. A similar construction will give an inverse of f, so f is a bijection. □

Proposition 2.5. *For all integers w, w', we have $\tau(w + w') \leq \tau(w) + \tau(w')$. Furthermore, equality holds if and only if $ww' = 0$ or $w, w' \in \Gamma$ or $-w, -w' \in \Gamma$.*

Proof. Consider the map $f : \Omega(w+w') \setminus \Omega(w) \to \Omega(w')$ given by $f(l) = l+w$. This is a well-defined injection. But $\Omega(w + w')$ is the disjoint union of $\Omega(w + w') \cap \Omega(w)$ and $\Omega(w + w') \setminus \Omega(w)$. Since $\Omega(w + w') \cap \Omega(w) \subseteq \Omega(w)$, we obviously have that $\tau(w + w') \leq \tau(w) + \tau(w')$. Furthermore, equality holds if and only if f is surjective and $\Omega(w) \subseteq \Omega(w + w')$. We show that $\tau(w + w') = \tau(w) + \tau(w')$ if and only if $w = 0$ or $w' = 0$ or $w, w' \in \Gamma$ or $-w, -w' \in \Gamma$: One implication is obvious, so let us assume that $\tau(w + w') = \tau(w) + \tau(w')$ and $w, w' \neq 0$. Then f is surjective and $\Omega(w) \subseteq \Omega(w + w')$ by the previous remark. If $w > 0$, then 0 is not in the image of f. But $im(f) = \Omega(w')$, and $0 \in \Omega(w')$ if and only if $w' \notin \Gamma$. Hence $w' \in \Gamma$. In particular $w' > 0$, so by interchanging the roles of w and w', we see that $w \in \Gamma$. If $w, w' < 0$, we get from proposition 2.4 that

$$\tau(-w) + \tau(-w') = \tau(w) + w + \tau(w') + w' = \tau(w + w') + w + w' = \tau(-w - w').$$

By the argument in the case $w > 0$, we get $-w, -w' \in \Gamma$. □

We remark that if $\Gamma = \mathbf{N_0}$, we have $\Omega(w) = \{0, 1, \dots, -(w+1)\}$ if w is negative, and $\Omega(w)$ is empty otherwise. So we obtain the formula $\tau(w) = 1/2(|w| - w)$ in this case. As a consequence, the two formulas above are trivial in the case $\Gamma = \mathbf{N_0}$.

4. The ring of differential operators

Let Γ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. We shall give an explicit description of the ring of differential operators $\mathrm{D}(A)$ on this monomial curve.

From corollary 2.3, we see that the set $\{P_w : w \in \mathbf{Z}\} \cup \{E\}$ generates $\mathrm{D}(A)$ as a k-algebra. It is also clear that $P_w P_{w'}$ and $P_{w+w'}$ are homogeneous differential operators of weight $w + w'$.

Lemma 2.6. *Let w, w' be integers. Then the homogeneous differential operators $P_w P_{w'}$ and $P_{w+w'}$ are equal if and only if one of the following conditions hold: $ww' = 0$, or $w, w' \in \Gamma$, or $w, w' \in -\Gamma$. Moreover, if none of these conditions holds, then $d(P_w P_{w'}) > d(P_{w+w'})$.*

Proof. From proposition 2.5, $\tau(w + w') = \tau(w) + \tau(w')$ if and only if one of the conditions $ww' = 0$, or $w, w' \in \Gamma$, or $w, w' \in -\Gamma$ is fulfilled. Furthermore, we have that $d(P_w P_{w'}) = d(P_w) + d(P_{w'})$ since A is an integral domain of finite type, and $d(P_w) = \tau(w)$, $d(P_{w'}) = \tau(w')$, and $d(P_{w+w'}) = \tau(w + w')$ from corollary 2.3. So $P_w P_{w'}$ and $P_{w+w'}$ have the same order if and only if one of the conditions $ww' = 0$, or $w, w' \in \Gamma$ or $w, w' \in -\Gamma$ is fulfilled. Moreover, if this is not the case, then $d(P_w P_{w'}) > d(P_{w+w'})$ from proposition 2.5. So it only remains to show that if the differential operators have the same order, they must be equal. But homogeneous differential operators in $\mathrm{D}(A)$ of weight $w + w'$ and order $\tau(w + w')$ can only differ by a scalar multiple in k^*: This is clear, since corollary 2.3 gives the dimension formula

$$\dim_k \mathrm{D}^p(A)_w = \begin{cases} p - \tau(w) + 1, & p \geq \tau(w) \\ 0, & p < \tau(w) \end{cases}$$

for all integers p, w. From corollary 2.3, we see that $P_w P_{w'}$ and $P_{w+w'}$ both have leading term $t^{w+w'+\tau(w+w')}\partial^{\tau(w+w')}$, so they must be equal. □

We see that any differential operator P_w with $w \in \Gamma$ can be written as a finite product of differential operators from the set $\{P_w : w = a_1, \dots, a_r\}$. Similarly, any differential operator P_w with $w \in -\Gamma$ can be written as a finite product of differential operators from the set $\{P_w : w = -a_1, \dots, -a_r\}$. It is therefore clear that the finite set G given by

$$G = \{P_w : |w| = a_1, \dots , a_r \text{ or } |w| \in H\} \cup \{E\}$$

is a set of generators for $\mathrm{D}(A)$ as k-algebra. However, it is not in general a minimal set of generators for $\mathrm{D}(A)$. See Smith and Stafford [**32**], section 3.12 for an easy counterexample: Let $\Gamma =< 2, 3 >$, then the generator P_{-3} is superfluous in the generating set G. In fact, $P_{-3} = 1/2\,[P_{-2}, P_{-1}]$ in this case.

If $\Gamma = \mathbf{N_0}$, then $G = \{t, \partial, E\}$, and the maximal order of a generator in G is 1. Assume that $\Gamma \neq \mathbf{N_0}$; we shall calculate the maximal order of the generators in G in this case as well: Obviously, $d(E) = 1$, $d(P_w) = 0$ for $w = a_1, \dots , a_r$, and by proposition 2.4, $d(P_w) = -w$ for $w = -a_1, \dots , -a_r$. If $w \in H$, then we have $\Omega(w) \subseteq \{0, 1, \dots , g - w\}$, and therefore $\tau(w) \leq g - w + 1 \leq g$. If $w \in -H$, then proposition 2.4 gives the inequality $\tau(w) = \tau(-w) - w \leq (g + w + 1) - w = g + 1 = c$. It follows that the maximal order of a differential operator in G is bounded by

$$\max\{d(P) : P \in G\} \leq \max\{a_r, c\}.$$

Moreover, $d(P_w) = a_r$ for $w = -a_r$, and $d(P_w) = \tau(-g) = \tau(g) + g = 1 + g = c$ for $w = -g$. So if $\Gamma \neq \mathbf{N_0}$, the equality

$$\max\{d(P) : P \in G\} = \max\{a_r, c\}$$

holds. Furthermore, if $\Gamma = \mathbf{N_0}$, then $\max\{a_r, c\} = 1$, so equality holds as well. We summarize these results in the following theorem:

Theorem 2.7. *Let Γ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. Then the k-algebra $\mathrm{D}(A)$ of differential operators on A*

has a finite generating set

$$G = \{P_w : |w| = a_1, \dots, a_r \text{ or } |w| \in H\} \cup \{E\}$$

of cardinality $2r+2h+1$. *Furthermore, the algebra* $\mathrm{D}(A)$ *is generated by the linear subspace* $\mathrm{D}^p(A) \subseteq \mathrm{D}(A)$ *with* $p = \max\{a_r, c\}$.

Since G is not a minimal generating set for $\mathrm{D}(A)$ as k-algebra, $p = \max\{a_r, c\}$ is in general not the minimal order such that $\mathrm{D}(A)$ is generated by $\mathrm{D}^p(A)$. In fact, in the example $\Gamma =< 2, 3 >$ mentioned above, $\mathrm{D}^p(A)$ generates $\mathrm{D}(A)$ for $p = 2$, while $a_r = 3$. However, in section 6, we shall show that the generating set G given above has nice properties when passing to the associated graded ring $\operatorname{gr} \mathrm{D}(A)$.

For all integers w, the characteristic polynomial $\chi_w \in k[\xi]$ has integer coefficients. Furthermore, E^n can be expressed in the form

$$E^n = t^n \partial^n + \sum_{i=1}^{n-1} c_{ni} t^i \partial^i,$$

with $c_{ni} \in \mathbf{N_0}$ for all natural numbers n and all indices i with $1 \le i \le n-1$. So the differential operator $P_w = t^w \chi_w(E)$ has integer coefficients c_i when written in standard form

$$P_w = t^{w+\tau(w)} \partial^{\tau(w)} + \sum_{i=0}^{\tau(w)-1} c_i t^{w+i} \partial^i,$$

and the coefficients c_i are easily computed from the numerical function τ and the expressions for powers of E given above. Furthermore, the leading term of P_w is $t^{w+\tau(w)} \partial^{\tau(w)} = t^{\tau(-w)} \partial^{\tau(w)}$ for all integers w.

Let A be any domain over k, and let $\overline{A}$ be its normalization. It turns out that it is very useful to compare the ring structures of $\mathrm{D}(A)$ and $\mathrm{D}(\overline{A})$, especially when A has Krull dimension 1:

Theorem 2.8. *Let A be an integral domain of finite type over k, such that A has Krull dimension* 1. *Then the following conditions are equivalent:*
i) The normalization map $\operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ *is injective.*
ii) $\mathrm{D}(A)$ *is a simple ring.*
iii) A is a simple left $\mathrm{D}(A)$*-module.*
iv) $\mathrm{D}(A)$ *is Morita equivalent to* $\mathrm{D}(\overline{A})$.

Proof. See Smith and Stafford [**32**], proposition 3.3, theorem 3.4, theorem 3.7, and proposition 4.2 and the following comments. □

Let us apply this theorem to the monomial curve $A = k[\Gamma]$: We know that the normalization of A is $\overline{A} = k[t]$, so $\mathrm{D}(\overline{A})$ is the Weyl algebra $A_1(k)$. Furthermore, the normalization map $\overline{X_\Gamma} \to X_\Gamma$ is bijective. So by theorem 2.8, we have that $\mathrm{D}(A)$ is a simple ring, Morita equivalent to the Weyl algebra $A_1(k)$, for all monomial curves $A = k[\Gamma]$.

Corollary 2.9. *Let Γ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. Then the ring* $\mathrm{D}(A)$ *is Morita equivalent to the Weyl algebra* $A_1(k)$. *Furthermore,* $\mathrm{D}(A)$ *has the following properties:*
i) $\mathrm{D}(A)$ *is a simple Noetherian ring.*
ii) A is a simple left $\mathrm{D}(A)$*-module.*
iii) $\mathrm{D}(A)$ *has Krull dimension* 1.
iv) $\mathrm{D}(A)$ *has Gelfand-Kirillov dimension* 2.
v) $\mathrm{D}(A)$ *is a hereditary ring.*

PROOF. From theorem 2.8, A is a simple left $\mathrm{D}(A)$-module. Since $\overline{A} = k[t]$ is a regular domain of finite type over k, the result holds for $\Gamma = \mathbf{N_0}$ by Smith and Stafford [**32**], section 1.4, and hence for $A = \overline{A}$. But in the general case, $\mathrm{D}(A)$ is Morita equivalent with $\mathrm{D}(\overline{A}) = A_1(k)$ by theorem 2.8. This means that there are equivalences of categories between left $\mathrm{D}(A)$-modules and left $A_1(k)$-modules, and between right $\mathrm{D}(A)$-modules and right $A_1(k)$-modules. But the property of being a simple Noetherian ring is a Morita equivalent property, and Krull dimension, Gelfand-Kirillov dimension and global dimension are Morita invariant numbers. This proves the corollary, since a ring is hereditary if and only if it has global dimension 1. □

We remark that the ring $\mathrm{D}(A)$ is neither left nor right Artinian: Let I_n be the left ideal in $\mathrm{D}(A)$ generated by t^n for $n \geq c$, and similarly, let J_n the right ideal in $\mathrm{D}(A)$ generated by t^n for $n \geq c$. Then $I_c \supseteq I_{c+1} \supseteq \dots$ is a chain of left ideals, and similarly $J_c \supseteq J_{c+1} \supseteq \dots$ is a chain of right ideals. But $I_n = I_{n+1}$ if and only if $Pt^{n+1} = t^n$ for some differential operator $P \in \mathrm{D}(A)$. Since A is an integral domain of finite type over k, we know that $d(P) + d(t^{n+1}) = d(t^n)$, so $d(P) = 0$ and $P \in A$. But we easily see, for instance by comparing degrees of polynomials in A, that this is a contradiction. So $\mathrm{D}(A)$ has an infinite descending chain of left ideals. Similarly, $J_n = J_{n+1}$ if and only if $t^{n+1}P = t^n$ for some differential operator $P \in \mathrm{D}(A)$, which is also impossible. So $\mathrm{D}(A)$ has an infinite descending chain of right ideals, as well.

5. The derivation module

Let Γ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. Then $\mathrm{Der}_k(A)$ is a $\mathbf{Z}$-graded left A-module in $\mathrm{D}(A)$, since A is of finite type. We shall give an explicit description of this A-module. It is well-known that if $\Gamma = \mathbf{N_0}$, then $\mathrm{Der}_k(A) = A\partial = k[t]\partial$, a free left A-module of rank 1. So this is a trivial case, and for the rest of this section, we shall assume that $\Gamma \neq \mathbf{N_0}$.

Let w be any integer. We shall denote by $\mathrm{Der}_k(A)_w$ the set of homogeneous derivations of A of weight w. By definition, $\mathrm{Der}_k(A)_w = \mathrm{D}(A)_w \cap \mathrm{Der}_k(A)$. We see that if $\tau(w) \leq 1$, then $\mathrm{Der}_k(A)_w$ is a one-dimensional k-vector space: If $w \in \Gamma$, it is generated by $t^w E = P_w E$, and if $\tau(w) = 1$, then it is generated by $t^w E = P_w$. Furthermore, $\mathrm{Der}_k(A)_w = 0$ if $\tau(w) \geq 2$.

Let us consider the set $\Gamma^{(1)} = \{w \in \mathbf{Z} : \tau(w) = 1\}$. We remark that $\Gamma^{(1)} \subseteq H$, and therefore $\Gamma^{(1)}$ is a finite set. To see this, assume that $w \in \Gamma^{(1)}$ for some strictly negative number w. If $w = -1$, then $\tau(-1) = \tau(1) + 1 \geq 2$, since $1 \notin \Gamma$. If $w \leq -2$, then $\tau(w) = \tau(-w) - w \geq 2$. So we can conclude that no strictly negative number w can have $\tau(w) = 1$. It follows that

$$\Gamma^{(1)} = \{w \in H : w + \gamma \in \Gamma \text{ for all non-zero } \gamma \in \Gamma\}.$$

Clearly, $\mathrm{Der}_k(A)$ is generated by E and $\{P_w : w \in \Gamma^{(1)}\}$ as a left A-module. So we immediately see that the subring of $\mathrm{D}(A)$ generated by A and $\mathrm{Der}_k(A)$ is positively graded. But $\mathrm{D}(A)$ is not positively graded: In fact, P_w is a non-zero homogeneous differential operator of weight w for all strictly negative numbers w. So for a monomial curve A, $\mathrm{D}(A)$ is generated by A and $\mathrm{Der}_k(A)$ if and only if A is regular. We remark that this proves Nakai's conjecture in the case of monomial curves.

Proposition 2.10. *Let $\Gamma \neq \mathbf{N_0}$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. Then, the set $\{t^w E : w \in \Gamma^{(1)} \text{ or } w = 0\}$ is a minimal set of homogeneous generators for $\mathrm{Der}_k(A)$ as left A-module.*

Proof. It is clear that this is a homogeneous set of generators of $\mathrm{Der}_k(A)$ as a left A-module. But let $w \in \Gamma^{(1)}$ or $w = 0$, and let $\gamma \in \Gamma$. Then $t^\gamma(t^w E) = t^{\gamma+w}E$, and $\gamma + w \in \Gamma$ if $\gamma \neq 0$, so none of the generators are superfluous. $\square$

Let A be any quasi-homogeneous k-algebra which is an integral domain of finite type over k. If A is Cohen-Macaulay, then the Cohen-Macaulay type of A is defined to be $t = t(A) = \dim_k Ext^1_A(k, A)$, where $k = A/A_+$, and A_+ is the unique graded maximal ideal $A_+ = \oplus_{i>0} A_i \subseteq A$. The Cohen-Macaulay type $t(A)$ of A is a strictly positive integer, and $t(A) = 1$ if and only if A is Gorenstein. See Bruns and Herzog [**8**] for a general reference on Cohen-Macaulay type.

It is well-known that any integral domain of Krull dimension 1 is Cohen-Macaulay. So the Cohen-Macaulay type of a monomial curve $A = k[\Gamma]$ is a well-defined natural number. By Fröberg [**14**], lemma 1, the Cohen-Macaulay type of a monomial curve $A = k[\Gamma]$ is the cardinality of the set $\Gamma^{(1)}$.

It is an elementary fact that $g \in \Gamma^{(1)}$, and that $\Gamma^{(1)} = \{g\}$ if and only if Γ is a symmetric semigroup. So in particular, we see that a monomial curve $A = k[\Gamma]$ is Gorenstein if and only if Γ is symmetric, which is a well-known result. More generally, the following result holds:

Proposition 2.11. *Let $\Gamma \neq \mathbf{N_0}$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. Then the minimal number of generators of the A-module $\mathrm{Der}_k(A)$ is given by $\mu(\mathrm{Der}_k(A)) = t(A) + 1$.*

All numerical semigroups of rank 2 are symmetric. So $t(A) = 1$, and $\{E, t^g E\}$ is a minimal set of generators for $\mathrm{Der}_k(A)$ in this case. Furthermore, it is known that there is no bound of $t(A)$ when Γ is a numerical semigroup of rank at least 4, see Cavaliere and Niesi [**10**], remark 3.3 and Fröberg, Gottlieb and Häggkvist [**13**], note 11. However, the situation for numerical semigroups of rank 3 is quite different: By Cavaliere and Niesi [**10**], proposition 3.2, the Cohen-Macaulay type $t(A) \leq 2$ in this case, and the possible values $t(A) = 1, 2$ are both obtained: $t(A) = 1$ for all symmetric numerical semigroups of rank 3, and $t(A) = 2$ for all non-symmetric numerical semigroups of rank 3. This gives a positive answer to a question raised by Skaar in her Master thesis [**31**], with applications to deformation theory of monomial curves:

Proposition 2.12. *Let Γ be a numerical semigroup of rank 3, and let $A = k[\Gamma]$ be the corresponding monomial curve. If Γ is symmetric, then $\{E, t^g E\}$ is a minimal set of homogeneous generators of $\mathrm{Der}_k(A)$ as left A-module. Otherwise, there exists a unique $h \neq g$ in $\Gamma^{(1)}$ such that $\{E, t^h E, t^g E\}$ is a minimal set of homogeneous generators of $\mathrm{Der}_k(A)$ as left A-module.*

6. The associated graded ring

Let Γ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. We shall give an explicit description of the associated graded ring $\mathrm{gr}\,\mathrm{D}(A)$ associated with this monomial curve.

Since $A \to T$ is an injective localization map of $\mathbf{Z}$-graded k-algebras, we shall consider $\mathrm{gr}\,\mathrm{D}(A)$ a $\mathbf{Z}^2$-graded subring of $\mathrm{gr}\,\mathrm{D}(T)$. But $T = k[t, t^{-1}]$, so let us fix the homogeneous coordinates $f : k[t, u] \to T$ for T given by $t \mapsto t$, $u \mapsto t^{-1}$. But $\partial_u \in \mathrm{D}(k[t, u])$ corresponds to $-t^2\partial \in \mathrm{D}(T)$, so the symbol relative to the coordinates f defines an injective homomorphism of $\mathbf{Z}^2$-graded k-algebras

$$\sigma : \mathrm{gr}\,\mathrm{D}(T) \to T[\xi],$$

where $\sigma(\partial) = \xi$. We recall that the differential operators in $\mathrm{D}(T)$ of order at most p are given as

$$\mathrm{D}^p(T) = \{\sum_{i=0}^{p} c_i \partial^i : c_i \in T\}$$

for all integers $p \geq 0$. Consequently, $\sigma : \operatorname{gr} \mathrm{D}(T) \to T[\xi]$ is surjective, and therefore an isomorphism. We shall consider $\operatorname{gr} \mathrm{D}(A)$ a $\mathbf{Z}^2$-graded subring of $T[\xi] = k[t, t^{-1}, \xi]$ via the symbol σ.

Let P be a non-zero differential operator in $\mathrm{D}(A)$. We may write P in standard form relative to the homogeneous coordinates f, since we identify P with a differential operator in $\mathrm{D}(T)$. Therefore, the symbol of P relative to the coordinates f is well-defined.

Assume that $p \geq \tau(w)$, and let $P = P_w E^{p-\tau(w)} \in \mathrm{D}(A)$. Then P is a homogeneous differential operator of weight w and order p, and we have

$$\sigma(P) = t^{w+p} \xi^p.$$

We know form corollary 2.3 that the set of differential operators $P_w E^{p-\tau(w)}$ for integers p, w such that $p \geq \tau(w)$ is a basis for $\mathrm{D}(A)$. Consequently, the set of monomials $t^{w+p}\xi^p$ for integers p, w such that $p \geq \tau(w)$ is a basis for $\operatorname{gr} \mathrm{D}(A)$ considered as a subring of $T[\xi]$.

Motivated by this fact, we shall introduce the new coordinates $(\alpha, \beta) \in \mathbf{Z}^2$ given by $\alpha = w + p$, $\beta = p$. Furthermore, we let $\Gamma' \in \mathbf{Z}^2$ be the subset consisting of all pairs $(\alpha, \beta) \in \mathbf{Z}^2$ such that $\beta \geq \tau(\alpha - \beta)$. Then $(\alpha, \beta) \in \Gamma'$ if and only if $t^\alpha \xi^\beta \in \operatorname{gr} \mathrm{D}(A)$ for integers $\alpha, \beta \in \mathbf{Z}^2$. In particular, $\Gamma' \subseteq \mathbf{Z}^2$ is a semigroup, and $\operatorname{gr} \mathrm{D}(A)$ is isomorphic to the semigroup algebra $k[\Gamma']$.

If $\Gamma = \mathbf{N_0}$, then $\Gamma' = \mathbf{N_0}^2$ and $\operatorname{gr} \mathrm{D}(A) \cong k[t, \xi]$, so this case is trivial. We assume that $\Gamma \neq \mathbf{N_0}$, and let $(\alpha, \beta) \in \Gamma'$ with $w = \alpha - \beta$. Then we have that $\tau(w) \leq \beta$, so let $m = \beta - \tau(w) \geq 0$, and put $(\alpha', \beta') = (\alpha, \beta) - m(1, 1)$. We see that $\beta' = \beta - m = \tau(w)$, and furthermore that

$$\tau(-w) = \tau(w) + w = \beta' + (\alpha - \beta) = \beta' + (\alpha' - \beta') = \alpha'.$$

Consequently, any $(\alpha, \beta) \in \Gamma'$ can be written uniquely in the form

$$(\alpha, \beta) = m(1, 1) + (\tau(-w), \tau(w))$$

with $m \in \mathbf{N_0}$, $w \in \mathbf{Z} \setminus \{0\}$. It follows that the finite set G' of cardinality $2r + 2h + 1$, given by

$$G' = \{(\tau(-w), \tau(w)) : |w| = a_1, \dots, a_r \text{ or } |w| \in H\} \cup (1, 1)$$

is a set of generators of Γ'. We claim that this is a minimal set of generators for Γ':

Lemma 2.13. *Let $\Gamma \neq \mathbf{N_0}$ be a numerical semigroup. Then the set G' is a minimal set of generators for Γ'.*

Proof. We have already seen that G' is a set of generators for Γ'. We show that none of the generators are superfluous: First, notice that $(1, 0), (0, 1) \notin \Gamma'$, so $(1, 1)$ is certainly not superfluous. Furthermore, $(\tau(-w), \tau(w))$ is given as $(a_i, 0)$ if $w = a_i$ and $(0, a_i)$ if $w = -a_i$. So by the minimality of the generator set $\{a_1, \dots, a_r\}$ for Γ, none of these generators are superfluous. Let w' be an integer such that $|w'| \in H$, and denote by I the set $I = \{w \in \mathbf{Z} \setminus \{w'\} : |w| \in H \text{ or } |w| = a_1, \dots, a_r\}$. We show that the generator $(\tau(-w'), \tau(w'))$ is not superfluous: Assume to the contrary that

$$(\tau(-w'), \tau(w')) = \sum_{w \in I} n_w (\tau(-w), \tau(w)) + m(1, 1).$$

Then $w' = \sum_{w \in I} n_w w$, and $\tau(w') = \tau(\sum_{w \in I} n_w w) = \sum_{w \in I} n_w \tau(w) + m$. By using proposition 2.5 repeatedly, we see that $m = 0$ and $\tau(\sum n_w w) = \sum n_w \tau(w)$.

Assume that $n_w \neq 0$ for some w with $|w| \in H$. Then, proposition 2.5 applied to the sum $w' = w + (w' - w)$ gives $w = w'$, which is impossible. But then $|w'| \in \Gamma$ by another application of proposition 2.5, and this contradicts the assumption that the generator $(\tau(-w'), \tau(w'))$ is superfluous. $\square$

We also notice that all generators in G' is in $\mathbf{N_0}^2 \subseteq \mathbf{Z}^2$, and that G' is invariant under the reflection in the line $\alpha = \beta$. We summarize these properties of the semigroup Γ' in the following proposition:

Proposition 2.14. *Let $\Gamma \neq \mathbf{N_0}$ be a numerical semigroup, and let $\Gamma' \subseteq \mathbf{Z}^2$ be the semigroup $\Gamma' = \{(\alpha, \beta) \in \mathbf{Z}^2 : \beta \geq \tau(\alpha - \beta)\}$. Then*

$$G' = \{(\tau(-w), \tau(w)) : |w| = a_1, \dots, a_r \text{ or } |w| \in H\} \cup (1,1)$$

is a minimal set of generators of Γ'. Furthermore, $\Gamma' \subseteq \mathbf{N_0}^2$, $\mathbf{N_0}^2 \setminus \Gamma'$ is a finite set, and for all integers α, β, $(\alpha, \beta) \in \Gamma'$ if and only if $(\beta, \alpha) \in \Gamma'$. In particular, $\Gamma' \subseteq \mathbf{Z}^2$ is an affine semigroup.

Proof. The only thing left to prove is the finiteness of $\mathbf{N_0}^2 \setminus \Gamma'$: For any integer w, consider the set $\{(\alpha, \beta) \in \Gamma' : \alpha - \beta = w\}$. This set has cardinality $\tau(w)$ if $w \geq 0$, so by the symmetry of Γ' it has cardinality $\tau(-w)$ if $w \leq 0$. From this, we get $|\mathbf{N_0}^2 \setminus \Gamma'| = 2 \sum_{w \in H} \tau(w)$. In particular, the set $\mathbf{N_0}^2 \setminus \Gamma'$ is finite (of even cardinality). $\square$

We remark that the following formula for the cardinality of $\mathbf{N_0}^2 \setminus \Gamma'$ was obtained by Jones in his thesis Jones [**17**]:

$$|\mathbf{N_0}^2 \setminus \Gamma'| = 2 \sum_{w \in H} w - h(h-1).$$

It is possible to give a much easier proof of this formula using our methods, but we will not include it here.

Corollary 2.15. *Let Γ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. Then $\operatorname{gr} \mathrm{D}(A)$ is the semigroup algebra $\operatorname{gr} \mathrm{D}(A) \cong k[\Gamma']$, where $\Gamma' \subseteq \mathbf{Z}^2$ is the affine semigroup $\Gamma' = \{(\alpha, \beta) \in \mathbf{Z}^2 : \beta \geq \tau(\alpha - \beta)\}$. If $\Gamma \neq \mathbf{N_0}$, then $\operatorname{gr} \mathrm{D}(A)$ has a minimal generating set*

$$\{t^{\tau(-w)} \xi^{\tau(w)} : |w| = a_1, \dots, a_r \text{ or } |w| \in H\} \cup \{t\xi\},$$

and if $\Gamma = \mathbf{N_0}$, it has a minimal generating set $\{t, \xi\}$. In particular, $\operatorname{gr} \mathrm{D}(A)$ is a subring of $\operatorname{gr} \mathrm{D}(\overline{A}) = k[t, \xi]$, and the k-algebra $\operatorname{gr} \mathrm{D}(A)$ has finite type over k and Krull dimension 2.

In contrast, we mention the following result of Smith and Stafford. Notice that in general, $\mathrm{D}(A)$ is not a subring of $\mathrm{D}(\overline{A})$. It is therefore rather surprising that $\operatorname{gr} \mathrm{D}(A) \subseteq \operatorname{gr} \mathrm{D}(\overline{A})$ for any affine curve.

Theorem 2.16. *Let A be a domain of finite type over k of Krull dimension 1. Then $\operatorname{gr} \mathrm{D}(A) \subseteq \operatorname{gr} \mathrm{D}(\overline{A})$. Furthermore, the following conditions are equivalent:*
i) The normalization map $\operatorname{Spec} \overline{X} \to \operatorname{Spec} X$ is injective.
ii) The k-algebra $\operatorname{gr} \mathrm{D}(A)$ is of finite type over k.
iii) The k-algebra $\operatorname{gr} \mathrm{D}(A)$ is Noetherian.

Proof. See Smith and Stafford [**32**], proposition 3.11 and theorem 3.12. $\square$

Let us consider the Bernstein filtration of $\mathrm{D}(A)$, as defined in section 6 of chapter 1, when A is a monomial curve. In this case, we see that there is a homogeneous, non-zero differential operator $P \in \mathrm{B}^n(A)$ of weight w if and only if $(n-w)/2 \geq \tau(w)$, or equivalently, if and only if $n \geq \tau(w) + \tau(-w)$. When $|w| \geq c$, where c is the conductor of Γ, then $\tau(w) + \tau(-w) = |w|$. In particular, we have $\mathrm{D}(A)_w \cap \mathrm{B}^n(A) = 0$ when $|w| > \max\{n, c\}$.

Corollary 2.17. *Let Γ be a numerical semigroup, and $A = k[\Gamma]$ the corresponding monomial curve. Then the k-vector space $\mathrm{B}^n(A)$ has finite dimension for all integers n.*

Proof. This is clear, since each $\mathrm{D}^p(A)_w$ has finite dimension by proposition 1.17, and furthermore each $\mathrm{B}^n(A)$ is the direct sum of a finite number of these k-vector spaces. □

We remark that $\operatorname{gr} \mathrm{D}(A)$ is a $\mathbf{Z}$-graded k-algebra via the Bernstein filtration. From the comment above, there is a non-zero homogeneous differential operator in $\mathrm{B}^n(A)$ of weight w if and only if $n \geq \tau(w) + \tau(-w)$. But it is easy to see that $\tau(w) + \tau(-w) \geq 0$, and equality holds if and only if $w = 0$. So we see that if $n < 0$, then $\mathrm{B}^n(A) = 0$, and if $n = 0$, then $\mathrm{B}^n(A) = k$. If follows that $\operatorname{gr} \mathrm{D}(A)$ is a quasi-homogeneous k-algebra of finite type.

7. An example

We finish this chapter by giving an example: Consider the numerical semigroup $\Gamma =< 3, 4 >$, and let $A = k[\Gamma] = k[t^3, t^4]$ be the affine coordinate ring of the corresponding monomial curve. We have that $r = 2,\ a_1 = 3,\ a_2 = 4,\ g = 5$, and $H = \{1, 2, 5\}$ in this case. We also notice that Γ is a symmetric, since $w \in \Gamma$ if and only if $5 - w \notin \Gamma$ for all integers $w \in \mathbf{Z}$.

Let $I = \{w \in \mathbf{Z} : |w| \in H \text{ or } |w| = a_1, \ldots, a_r\}$. We calculate the sets $\Omega(w)$ for $w \in I$, and obtain the set G of generators for $\mathrm{D}(A)$ as a k-algebra, given by $G = \{P_w : w \in I\} \cup \{P_0\}$ with

$$
\begin{aligned}
P_0 &= E \\
P_1 &= tE(E-4) = t^3\partial^2 - 3t^2\partial \\
P_2 &= t^2E(E-3) = t^4\partial^2 - 2t^3\partial \\
P_3 &= t^3 \\
P_4 &= t^4 \\
P_5 &= t^5E = t^6\partial \\
P_{-1} &= t^{-1}E(E-3)(E-6) = t^2\partial^3 - 6t\partial^2 + 10\partial \\
P_{-2} &= t^{-2}E(E-3)(E-4)(E-7) = t^2\partial^4 - 8t\partial^3 + 26\partial^2 - 36t^{-1}\partial \\
P_{-3} &= t^{-3}E(E-4)(E-8) = \partial^3 - 9t^{-1}\partial^2 + 21t^{-2}\partial \\
P_{-4} &= t^{-4}E(E-3)(E-6)(E-9) = \partial^4 - 12t^{-1}\partial^3 + 52t^{-2}\partial^2 - 80t^{-3}\partial \\
P_{-5} &= t^{-5}E(E-3)(E-4)(E-6)(E-7)(E-10) \\
&= t\partial^6 - 15\partial^5 + 110t^{-1}\partial^4 - 490t^{-2}\partial^3 + 1300t^{-3}\partial^2 - 1620t^{-4}\partial.
\end{aligned}
$$

We also see that $\mathrm{Der}_k(A)$ has a minimal generating set $\{E, t^5E\} = \{t\partial, t^6\partial\}$ as a left A-module.

We may now easily read off the minimal generating set g of $\operatorname{gr} \mathrm{D}(A)$ as a k-algebra. This is given by $\{p_w : w \in I\} \cup \{p_0\}$, with

$$
\begin{aligned}
&p_0 = tu, \\
&p_1 = t^3u^2, \quad p_2 = t^4u^2, \quad p_3 = t^3, \quad p_4 = t^4, \quad p_5 = t^6u, \\
&p_{-1} = t^2u^3, \quad p_{-2} = t^2u^4, \quad p_{-3} = u^3, \quad p_{-4} = u^4, \quad p_{-5} = tu^6
\end{aligned}
$$

Equivalently, $\operatorname{gr} \mathrm{D}(A) = k[\Gamma']$, where Γ' is the affine semigroup $\Gamma' = \mathbf{N_0}^2 \setminus H'$, where H' is the finite set given by

$$H' = \{(0,1), (0,2), (0,5), (1,0), (1,2), (1,3), (2,0), (2,1), (3,1), (5,0)\}.$$

CHAPTER 3

D-modules on monomial curves

In this chapter, we study D-modules on monomial curves. We introduce dimension and multiplicity for D-modules on monomial curves, and define the category of holonomic D-modules. In order to do this, we need a theory of Hilbert functions on graded modules over a quasi-homogeneous k-algebra, which we develop in section 2. We also study certain localization functors, defined on the category of D-modules and the category of graded D-modules.

1. Basic properties of D-modules

Let A be a commutative k-algebra, and let $\mathrm{D}(A)$ be the ring of differential operators on A. We shall often suppress the ring A from the notation, when no confusion is likely to arise from this, and write D for the ring $\mathrm{D}(A)$. We denote by a *D-module* any left $\mathrm{D}(A)$-module M. Furthermore, we denote by D-**Mod** the category of D-modules, where the morphisms are left D-module homomorphisms.

We have inclusions $k \subseteq A \subseteq D$ of commutative subrings of D. We see that k is in the centre of D, whereas A is usually not in this centre. It will often be convenient to consider a D-module as an A-module or as a vector space over k via the obvious forgetful functors.

We define the category $\mathbf{MC}(A;D)$ in the following way: An object in this category is a pair (M,ρ), where M is an A-module and $\rho : D \to \mathrm{D}(M)$ is a homomorphism of filtered k-algebras. A morphism in this category from the object represented by (M,ρ) to the object represented by (M',ρ') is a homomorphism $\phi : M \to M'$ of A-modules, such that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{D}(M) \arrow[dr, "\mathrm{D}(M,\phi)"] & \\ D \arrow[ur, "\rho"] \arrow[dr, "\rho'"'] & & \mathrm{D}(M,M') \\ & \mathrm{D}(M') \arrow[ur, "\mathrm{D}(\phi,M')"'] & \end{tikzcd}$$

Clearly, there is a functor $F : \mathbf{MC}(A;D) \to$ D-**Mod**: For any object (M,ρ) in $\mathbf{MC}(A;D)$, we may consider M as a k-vector space, with a D-module structure given by $Pm = \rho(P)(m)$ for all $P \in D,\ m \in M$. Furthermore, the commutative diagram above implies that any morphism in the category $\mathbf{MC}(A;D)$ is a morphism of the corresponding D-modules. We shall see that the functor F is in fact an equivalence of categories between $\mathbf{MC}(A;D)$ and D-**Mod**:

Proposition 3.1. *There is a natural equivalence of categories between* D-**Mod** *and* $\mathbf{MC}(A;D)$.

Proof. Let M be any D-module. Then we may consider M as a k-vector space, and the D-module structure is given by a homomorphism $\rho : D \to \mathrm{End}_k(M)$ of k-algebras. So it is clearly enough to show that the image of ρ is in $\mathrm{D}(M)$ when

M is considered as an A-module, and that the induced morphism $\rho : D \to \mathrm{D}(M)$ is a morphism of filtered k-algebras. We show this by induction: First, it is clear that $\rho(\mathrm{D}^p(A)) \subseteq \mathrm{D}^p(M)$ when $p = 0$ (and indeed for $p < 0$). So assume that the inclusion holds for some integer $p \geq 0$. Let $P \in \mathrm{D}^{p+1}(A)$ be a differential operator, then we have

$$[\rho(P), a] = [\rho(P), \rho(a)] = \rho([P, a])$$

for all $a \in A$. But $[P, a] \in \mathrm{D}^p(M)$, so $[\rho(P), a] \in \mathrm{D}^p(M)$ for all $a \in A$ by the induction hypothesis. By definition, this means that $\rho(P) \in \mathrm{D}^{p+1}(M)$. So $\rho(\mathrm{D}^{p+1}(A)) \subseteq \mathrm{D}^{p+1}(M)$, and this concludes the induction argument. □

We define the *derivation ring* of A to be the subring $\Delta(A) \subseteq D$ generated by $\mathrm{D}^1(A) = A \oplus \mathrm{Der}_k(A)$. This is a filtered subring of D in a natural way, having A as a subring. If A is a regular integral domain of finite type over k, then $\Delta(A) = D$. It has been conjectured by Nakai that if A is an integral domain of finite type over k, then $\Delta(A) = D$ if and only if A is regular. This conjecture has been proven when $\dim A = 1$, see Mount and Villamayor [**25**].

There are several approaches to the study of D-modules. The classical approach has been to take the commutative point of view, and study the category $\mathbf{MC}(A; D)$. When A is a regular k-algebra of finite type, this is the same as the study of pairs (M, ∇), where M is a module over the commutative ring A, and ∇ is an integrable connection of M (or equivalently, an integrable covariant derivative on M). In chapter 4, we shall consider modules with integrable covariant derivatives when A is not necessarily regular, which will lead to a theory equivalent to the theory of $\Delta(A)$-modules.

However, the studies of D-modules undertaken by considering the category $\mathbf{MC}(A; D)$ tend to hide the non-commutative aspect of D-module theory. Furthermore, since the notion of a D-module of finite type does not seem to have a natural counterpart in the category $\mathbf{MC}(A; D)$, such studies also tend to concentrate the attention on D-modules which are of finite type considered as A-modules.

In the rest of this chapter, we shall study D-modules more directly, taking an essentially non-commutative point of view. The main tools will be the use of invariants, and a certain localization process to be described later.

2. Hilbert functions on quasi-homogeneous k-algebras

The purpose of this section is to develop a theory for the Hilbert function of a positively graded module M over a quasi-homogeneous k-algebra A, generalizing the well-known theory in the case when A is generated by homogeneous elements of degree 1. The paper Campbell, Geramita, Hughes, Smith and Wehlau [**9**], from now on only called CGHSW [**9**], covers the case $M = A$. It has also inspired the work in this section, which does not appear in the literature, as far as we know. We shall refer to chapter 4 in Bruns and Herzog [**8**] for the basic results in this section.

Let A be a quasi-homogeneous, commutative k-algebra of finite type, and let M be a positively graded A-module of finite type. Then $A_0 = k$, and M_n is an A_0-module of finite type for all $n \geq 0$. The dimension $\dim_k M_n$ is therefore finite for all integers $n \geq 0$, and we see that it is invariant under isomorphisms of graded A-modules. We denote by $\mathrm{H}(M, -)$ the *Hilbert function* of M, given by $\mathrm{H}(M, n) = \dim_k M_n$ for all $n \geq 0$. Furthermore, we denote by $\mathrm{H}_M(t)$ the *Hilbert series* of M, defined as the power series $\mathrm{H}_M(t) = \sum \mathrm{H}(M, n)\, t^n$ generated by the Hilbert function. For the rest of this section, we shall assume that $M \neq 0$, and we shall denote the Krull dimension of M by $d = \dim M = \dim(A / \operatorname{ann} M)$.

Theorem 3.2. *Let A be a quasi-homogeneous k-algebra of finite type, and let M be a non-zero, positively graded A-module of finite type with $d = \dim M$. Then there exists an integer $l \geq 1$ and a polynomial $Q(t) \in \mathbf{Z}[t]$ with $Q(1) > 0$, such that*

$$\mathrm{H}_M(t) = \frac{Q(t)}{(1-t^l)^d}$$

Proof. Let $x_1, \ldots, x_d$ be a homogeneous system of parameters for M, of strictly positive degrees $n_1, \ldots, n_d$, and let $l = \mathrm{lcm}(n_1, \ldots, n_d)$. Then

$$x_1^{l/n_1}, \ldots, x_d^{l/n_d}$$

is another homogeneous system of parameters for M. Since these parameters all have degree l, the result follows from Bruns and Herzog [**8**], proposition 4.4.1 and the following remarks. □

We notice that l may be chosen as the least common multiple of the degrees of any homogeneous system of parameters for M. If it is, let us denote the corresponding graded Noether normalization of M by S. Then it follows from Bruns and Herzog [**8**], lemma 4.1.13 that $Q(1) = \mathrm{rk}_S M > 0$. But there may exist values for l which do not occur in this way. For instance, consider the example $A = k[t^2, t^3]$, $M = A$. In this case, we may choose $l = 1$. But clearly, there is no homogeneous system of parameters of degree 1, since $A_1 = 0$.

We let $L(M) = \{l \in \mathbf{N_0} : \mathrm{H}_M(t)\ (1-t^l)^d \in \mathbf{Z}[t],\ l \geq 1\}$, and define the *period* of M to be $m = m(M) = \min L(M)$. This is an invariant of M, and by CGHSW [**9**], lemma 2.1, we have that $L(M) = \{l \in \mathbf{N_0} : m \mid l,\ l \geq 1\}$. Let us define the polynomial $Q_M(t) \in \mathbf{Z}[t]$ to be $Q_M(t) = \mathrm{H}_M(t)\ (1-t^m)^d$. Since there exists some $l \in L(M)$ such that the corresponding polynomial $Q'(t) = \mathrm{H}_M(t)\ (1-t^l)^d$ satisfies $Q'(1) > 0$, and this l is a multiple of m from the description of $L(M)$, we see that $Q_M(1) > 0$ as well.

Proposition 3.3. *Let $H(t) \in \mathbf{Z}[[t]]$ be any non-zero power series $H(t) = \sum h_n t^n$ with non-negative coefficients $h_n \geq 0$, and let l, s be non-negative integers with $l \geq 1$. Denote by $a(t)$ the power series $a(t) = H(t)\ (1-t^l)^s$. Then the following conditions are equivalent:*

i) $a(t)$ is a polynomial in $\mathbf{Z}[t]$.

ii) There exist polynomials $P_r(t) \in \mathbf{Q}[t]$ for $0 \leq r < l$, such that $\deg P_r(t) \leq s-1$ and $P_r(n) = h_{nl+r}$ for all $n >> 0$.

Furthermore, if the conditions above hold, then $a(1) > 0$ if and only if $P_r(t)$ has degree $s-1$ for at least one integer r with $0 \leq r < l$.

Proof. See CGHSW [**9**], proposition 2.3 and remark 2.4. □

Assume that the conditions of proposition 3.3 hold. In this case, we follow the notation of the proof, and write $a_r(t)$ for the polynomial $a_r(t) = \sum \alpha_{nl+r} t^n \in \mathbf{Z}[t]$, where the coefficients α_i for $i \geq 0$ are given by $a(t) = \sum \alpha_i t^i$. So we obtain polynomials $a_0(t), \ldots, a_{l-1}(t) \in \mathbf{Z}[t]$, and we have $a(1) = a_0(1) + \cdots + a_{l-1}(1)$. Furthermore, the polynomial $P_r(t)$ has the form

$$P_r(t) = \frac{a_r(1)}{(s-1)!}\ t^{s-1} + P'_r(t)$$

for $0 \leq r < l$, where $P'_r(t)$ is a polynomial in $\mathbf{Q}[t]$ of degree $\deg P'_r(t) < s-1$. Consequently, we see that $a_r(1) \geq 0$ for all r, and that $\deg P_r(t) = s-1$ if and only if $a_r(1) > 0$.

Let us apply proposition 3.3 to the Hilbert series $\mathrm{H}_M(t)$: We let $l = m$, $s = d$ and $H(t) = \mathrm{H}_M(t)$. Then the first condition of proposition 3.3 is satisfied, with $a(t) = Q_M(t)$ and $a(1) = Q_M(1) > 0$. So let $e_r = e_r(M) = a_r(1)$ for $0 \leq r < m$.

Then $e_r \geq 0$ is a positive integer for $0 \leq r < m$, and $\sum e_r = Q_M(1) > 0$. We obtain the following result on the Hilbert function of M:

Corollary 3.4. *Let A be a quasi-homogeneous k-algebra of finite type, and let M be a non-zero, positively graded A-module of finite type with $d = \dim M$. Then there exist polynomials $p_M^r(t) \in \mathbf{Q}[t]$ for $0 \leq r < m$, such that $\deg p_M^r(t) \leq d-1$, and $p_M^r(n) = \mathrm{H}_M(nm+r)$ for all $n >> 0$. Furthermore, $e_r > 0$ for at least one integer r with $0 \leq r < m$, and if $e_r > 0$, then $p_M^r(t)$ has leading term $\frac{e_r}{(d-1)!}\, t^{d-1}$.*

A function $f : \mathbf{N_0} \to \mathbf{N_0}$ is a *quasi-polynomial* of period l if it is given by l polynomials $p_0(t), \ldots, p_{l-1}(t) \in \mathbf{Q}[t]$, in such a way that $f(nl+r) = p_r(n)$ for $0 \leq r < l$, $n \geq 0$, and l is the least positive integer with this property. We see that there exists a unique quasi-polynomial p_M of period m, defined by the polynomials $p_M^0(t), \ldots, p_M^{m-1}(t)$ given above, such that $p_M(n) = \mathrm{H}(M,n)$ for all $n >> 0$. We denote this quasi-polynomial p_M, and call it the *Hilbert quasi-polynomial* of M.

In fact, the polynomials $p_M^0(t), \ldots, p_M^{m-1}(t)$ are themselves Hilbert polynomials. We introduce the *degree modules*, defined in the following way: For $0 \leq r < m$, we define $A[m;r] = \sum A_{nm+r}$ and $M[m;r] = \sum M_{nm+r}$. We say that $a \in A[m;r]$ is homogeneous of degree n if $a \in A_{nm+r}$, and correspondingly that $m \in M[m;r]$ is homogeneous of degree n if $m \in M_{nm+r}$. If follows that $A[m;0]$ is a quasi-homogeneous k-algebra, and that $A[m;r], M[m;r]$ are positively graded $A[m;0]$-modules for $0 \leq r < m$.

Lemma 3.5. *Let A be a quasi-homogeneous k-algebra of finite type, and let M be a non-zero, positively graded A-module of finite type with $d = \dim M$. Then we have:*

i) $A[m;0]$ is a k-algebra of finite type, and $\dim A[m;0] = \dim A$.

ii) M is a $A[m;0]$-module of finite type, and it has Krull dimension d considered as an $A[m;0]$-module.

Proof. The first part is contained in CGHSW [**9**], proposition 4.2, which shows that A is an integral ring extension of $A[m;0]$. For the second part, it is clear that M is of finite type over $A[m;0]$, since A is integral over $A[m;0]$. So it is enough to notice that the ring extension of quotients $A[m;0]_M \subseteq A_M$ is integral as well (where we define $A_M = A/ann_A(M)$ for any A-module M). □

From lemma 3.5, we see that $M[m;r]$ is an $A[m;0]$-module of finite type for $0 \leq r < m$. Furthermore, corollary 3.4 shows that $p_M^r(t)$ is the Hilbert polynomial of $M[m;r]$. Let d_r denote the Krull dimension of $M[m;r]$ for all integers r with $0 \leq r < m$ such that $M[m;r] \neq 0$. Then $d_r \leq d$ for all r, and $d_r = d$ for at least one integer r. Even though $A[m;0]$ is not, in general, generated by homogeneous elements of degree 1, we have the following result:

Lemma 3.6. *Let r be an integer with $0 \leq r < m$. Then, the $A[m;0]$-module $M[m;r]$ has Hilbert polynomial $p_M^r(t)$, and $\deg p_M^r(t) = d_r - 1$ if $M[m;r] \neq 0$.*

Proof. The first part is clear. For the last part, let r be some integer such that $M[m;r] \neq 0$. We denote by $p(t)$ the Hilbert series of $M[m;r]$, and by s the degree of the polynomial $p_M^r(t)$. Since $p_M^r(t)$ is the Hilbert polynomial of $M[m;r]$, it follows from proposition 3.3 that $a(t) = p(t)\,(1-t)^{s+1}$ is a polynomial in $\mathbf{Z}[t]$ with $a(1) > 0$. But from theorem 3.2, we see that there is an integer $l \geq 1$ such that $b(t) = p(t)(1-t^l)^{d_r}$ is another polynomial in $\mathbf{Z}[t]$ with $b(1) > 0$. So we have the polynomial identity $a(t)(1-t^l)^{d_r} = b(t)(1-t)^{s+1}$, and $s = d_r - 1$ by counting the multiplicity of the irreducible factor $(t-1)$ on each side of this identity. □

We define the higher iterated Hilbert functions $\mathrm{H}^i(M,-)$ for $i \geq 0$ in the following way: Let $\mathrm{H}^0(M,-)$ be given by $\mathrm{H}^0(M,n) = \mathrm{H}(M,n)$, and let $\mathrm{H}^i(M,-)$

be given inductively by

$$\mathrm{H}^i(M,n) = \sum_{j \le n} \mathrm{H}^{i-1}(M,j)$$

for $i > 0$. The following observation is implicit in Bruns and Herzog [**8**], lemma 4.1.2: For any integer $i \ge 0$, there is a polynomial $a(t) \in \mathbf{Q}[t]$ of degree s such that $a(n) = \mathrm{H}^i(M,n)$ for all $n >> 0$ if and only if there is a polynomial $b(t) \in \mathbf{Q}[t]$ of degree $s+1$ such that $b(n) = \mathrm{H}^{i+1}(M,n)$ for all $n >> 0$. Furthermore, if there are such polynomials, then $a(t)$ has leading term $\frac{c}{s!}\, t^s$ if and only if $b(t)$ has leading term $\frac{c}{(s+1)!}\, t^{s+1}$. We use the fact that similar properties hold for quasi-polynomials:

Theorem 3.7. *Let A be a quasi-homogeneous k-algebra of finite type, and let M be a non-zero, positively graded A-module with $d = \dim M$. Then there exist polynomials $P^r_M(t) \in \mathbf{Q}[t]$ for $0 \le r < m$, such that $P^r_M(n) = \mathrm{H}^1(M, nm+r)$ for all $n >> 0$. Furthermore, the leading term of the polynomial $P^r_M(t)$ is $\frac{e}{d!} t^d$ for $0 \le r < m$, where $e = e_0 + \cdots + e_{m-1}$.*

Proof. If $d = 0$, then $\dim_k M$ is finite by theorem 3.2. So $\mathrm{H}^1(nm+r) = \dim_k M$ for all $n >> 0$ and all integers r with $0 \le r < m$, and $p^r_M(t) = e_r = \dim_k M[m;r]$. We may therefore assume that $d > 0$. Let us fix an integer r with $0 \le r < m$, and calculate $\mathrm{H}^1(M, nm+r)$ using the induction formula for $\mathrm{H}^1(M, -)$:

$$\begin{aligned} \mathrm{H}^1(M, nm+r) &= \sum_{j \le nm+r} \mathrm{H}(M,j) \\ &= \sum_{s=0}^{r} \sum_{i=0}^{n} \mathrm{H}(M, im+s) + \sum_{s=r+1}^{m-1} \sum_{i=0}^{n-1} \mathrm{H}(M, im+s). \end{aligned}$$

Let us fix an integer s with $0 \le s < m$. Then, we have that $\mathrm{H}(M, im+s) = p^s_M(i)$ for all $i >> 0$, and we obtain the formula

$$\sum_{i=0}^{n} \mathrm{H}(M, im+s) = \sum_{i=0}^{n} p^s_M(i) + D_s$$

for all $n >> 0$, where D_s is an integer constant that only depends on s. Since $p^s_M(t) = \frac{e_s}{(d-1)!} t^{d-1} + p'_s(t)$ with $\deg p'_s(t) < d-1$, the observation preceeding the theorem implies the following result: For all integers s with $0 \le s < m$, there is a polynomial $P_s(t) \in \mathbf{Q}[t]$ of the form $P_s(t) = \frac{e_s}{d!} t^d + P'_s(t)$ with $\deg P'_s(t) < d$, such that $P_s(n) = \sum_{i \le n} \mathrm{H}(M, im+s)$ for all $n >> 0$. Consequently, we have the formula

$$\mathrm{H}^1(M, nm+r) = \sum_{0 \le s \le r} P_s(n) + \sum_{r < s < m} P_s(n-1)$$

for all $n >> 0$. So let $P^r_M(t) = P_0(t) + \cdots + P_r(t) + P_{r+1}(t-1) + \cdots + P_{m-1}(t-1)$. Then $P^r_M(t) \in \mathbf{Q}[t]$ is a polynomial of degree d and with leading term $\frac{e}{d!} t^d$, where $e = e_0 + \cdots + e_{m-1}$. But $P^r_M(n) = \mathrm{H}^1(M, nm+r)$ for all $n >> 0$, which concludes the proof of the theorem. $\square$

We define the *first iterated Hilbert quasi-polynomial* of M to be the quasi-polynomial P_M of period m defined by the polynomials $P^0_M(t), \ldots, P^{m-1}_M(t)$. From theorem 3.7, it follows that all the polynomial components $P^r_M(t)$ of this quasi-polynomial have the same leading term $\frac{e}{d!} t^d$ with $e = e_0 + \cdots + e_{m-1}$. We define *the multiplicity* of M to be $e = e(M) = e_0 + \cdots + e_{m-1}$, which is clearly an invariant of M. We see that $e(M)$ is a strictly positive integer if $M \neq 0$. Furthermore, the cardinality of the set $\{r : e_r \neq 0\} = \{r : d_r = d\}$ is a lower bound for $e(M)$.

Finally, the following remark is in order: Let M be any graded module over a quasi-homogeneous k-algebra A. If M is of finite type, there exists some integer n

such that $M_i = 0$ for all $i < n$: This is clear, it is enough to let n be the minimal degree of a finite set of homogeneous generators of M. But then the twisted A-module $M(n)$ is positively generated, and the Hilbert function of M is related to the Hilbert function of $M(n)$ by the equation $\mathrm{H}(M,i) = \mathrm{H}(M(n), i-n)$. So the condition that M is positively graded, imposed in this section, is not essential.

3. Dimension and multiplicity

Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let D be the ring of differential operators on A. We recall that the Bernstein filtration $\{\mathrm{B}^n(A)\}$ is an exhaustive, ascending filtration of the k-algebra D, and that the associated graded ring $\mathrm{gr}\, D$ is a $\mathbf{Z}$-graded k-algebra via the Bernstein filtration. Notice that $\mathrm{gr}\, D$ is a quasi-homogeneous k-algebra of finite type.

Let M be a D-module. We denote by a *filtration* of M any ascending, exhaustive filtration $\{M_n : n \in \mathbf{Z}\}$ of M compatible with the Bernstein filtration of D, such that $M_n = 0$ for $n < 0$ and M_n is a finite dimensional vector space over k for all integers $n \geq 0$. For any filtration (M_n) of M, we consider the associated graded $\mathrm{gr}\, D$-module $\mathrm{gr}\, M = \oplus M_n / M_{n-1}$. This is a positively graded $\mathrm{gr}\, D$-module, and it will of course depend upon the chosen filtration. We say that the filtration (M_n) is a *good filtration* if $\mathrm{gr}\, M$ is a $\mathrm{gr}\, D$-module of finite type. From Björk [**3**], proposition 2.6.1, we see that there exists a good filtration of a D-module M if and only if M is of finite type over D.

We shall only be interested in D-modules of finite type. From now on, a D-module will therefore denote a D-module of finite type. Consequently, all D-modules have a good filtration. A good filtration is usually not unique, but we have the following result:

Lemma 3.8. *Let M be a D-module. If $(M_n), (M'_n)$ are two good filtrations of M, then there exists a integer $p \geq 0$ such that $M'_{n-p} \subseteq M_n \subseteq M'_{n+p}$ for all integers n.*

Proof. See Björk [**3**], proposition 2.6.1. □

Let M be a non-zero D-module, and let (M_n) be any good filtration of M. Then the associated graded module $\mathrm{gr}\, M$ is non-zero, so the conditions of theorem 3.7 is satisfied for the $\mathrm{gr}\, D$-module $\mathrm{gr}\, M$. Consequently, there exists a quasi-polynomial $P : \mathbf{N_0} \to \mathbf{N_0}$ such that $\dim_k M_n = P(n)$ for all $n >> 0$. Let $d = \dim \mathrm{gr}\, M$ and let e be the multiplicity of $\mathrm{gr}\, M$. We see from lemma 3.8 that d, e are independent upon the good filtration (M_n). We define the *dimension* of the D-module M to be $d = d(M) = \dim \mathrm{gr}\, M$, and the *multiplicity* $e = e(M)$ of M to be the multiplicity of $\mathrm{gr}\, M$. We see that the dimension and multiplicity of M are well-defined, non-negative integers with $d \geq 0,\ e \geq 1$. We also remark that $d = 0$ if and only if M is a finite dimensional vector space over k, see theorem 3.2.

Consider the example $M = D$. Clearly, the Bernstein filtration of D is a good filtration, and $\dim_k \mathrm{B}^n(A) = \frac{1}{2}(n+1)(n+2) - r$ for $n >> 0$, where r is the cardinality of the finite set ${\mathbf{N_0}}^2 \setminus \Gamma'$. Consequently, we have $d(D) = 2$ and $e(D) = 1$.

Proposition 3.9. *Let $0 \to N \to M \to P \to 0$ be an exact sequence of non-zero D-modules. Then we have:*
i) $d(M) = \max\{d(N), d(P)\}$.
ii) If $d(N) = d(P)$, then $e(M) = e(N) + e(P)$.
iii) If $d(N) > d(P)$, then $e(M) = e(N)$, and if $d(P) > d(N)$, then $e(M) = e(P)$.

Proof. Denote the morphisms $f : N \to M$ and $g : M \to P$, and let (M_n) be a good filtration of M. Then there are induced filtrations (N_n) of N and (P_n) of P, given by $N_n = f^{-1}(M_n)$ and $P_n = g(M_n)$ for all integers n. We obtain an exact sequence of $\mathrm{gr}\, D$-modules $0 \to \mathrm{gr}\, N \to \mathrm{gr}\, M \to \mathrm{gr}\, P \to 0$, and $\mathrm{gr}\, D$ is a Noetherian ring. Since (M_n) is a good filtration, the same holds for (N_n) and (P_n). But $\dim_k M_n/M_{n-1} = \dim_k N_n/N_{n-1} + \dim_k P_n/P_{n-1}$ for all $n \geq 0$, and $M_n = N_n = P_n = 0$ for $n < 0$, so $\dim_k M_n = \dim_k N_n + \dim_k P_n$ for all $n \geq 0$. Consequently, $P_M(n) = P_N(n) + P_P(n)$ for all $n \geq 0$, and the result follows. □

Corollary 3.10. *Let $f : M \to M'$ be an isomorphism of non-zero D-modules. Then $d(M) = d(M')$ and $e(M) = e(M')$. In particular, the dimension $d(M)$ and the multiplicity $e(M)$ of a D-module M are invariants of M.*

Proof. From the proof of proposition 3.9, we see that $P_M(n) = P_{M'}(n)$ for all $n \geq 0$. The rest is clear. □

Let M be a non-zero D-module, then there is a surjection $D^n \to M$ of D-modules for some $n \geq 0$. Since $d(D) = 2$, $e(D) = 1$, we see from proposition 3.9 that $d(D^n) = 2$ and $e(D^n) = n$. If follows that $d(M) \leq d(D^n) = 2$ for any D-module. A more interesting fact is that there is a lower bound for the dimension of D-modules. This lower bound is classically known as *Bernstein's inequality*:

Proposition 3.11. *Let M be a non-zero D-module. Then $d(M) \geq \dim A = 1$.*

Proof. Assume that M is non-zero and $d(M) = 0$. Then M is of finite type as vector space over k. By corollary 2.9, D is a simple ring. So the ring homomorphism $D \to \mathrm{End}_k(M)$ defining the D-module structure on M is an injective ring homomorphism, since $M \neq 0$. But $\mathrm{End}_k(M)$ is a finite dimensional vector space over k, and D is clearly not of finite dimension as a vector space over k, so this is a contradiction. □

We say that a D-module M is *holonomic* if $M = 0$ or if $M \neq 0$ and $d(M) = 1$. Let $0 \to N \to M \to P \to 0$ be an exact sequence of D-modules. Then M is holonomic if and only if N and P are holonomic: This is clear from proposition 3.9. So in particular, submodules and quotients of holonomic modules are holonomic, and extensions of holonomic modules are holonomic. We also see that any finite sum of holonomic modules is holonomic: This is clear, since such a sum is a quotient of a direct sum of holonomic modules.

Lemma 3.12. *Let $I \subseteq D$ be a non-zero left ideal in D. Then D/I is a holonomic D-module.*

Proof. Assume that I is generated by a non-zero operator $P \in D$. Then we have an exact sequence of D-modules $0 \to D \to D \to D/I \to 0$ induced by the right multiplication with P on D. If $d(D/I) = 2$, then $e(D) = e(D) + e(D/I)$ by proposition 3.9. This means that $e(D/I) = 0$, which is a contradiction. So $d(D/I) = 1$ or $D/I = 0$, and D/I is holonomic. For the general case, assume that $I \subseteq D$ is a non-zero left ideal, and choose a non-zero operator $P \in I$. Let J be the left ideal $J = DP \subseteq I$, and consider the short exact sequence

$$0 \to I/J \to D/J \to D/I \to 0.$$

We know that D/J is holonomic, so D/I is a quotient of a holonomic module, and hence holonomic. □

Proposition 3.13. *Let M be a D-module. Then M has finite lenght if and only if M is Artinian, and the following conditions are equivalent:*
i) M is holonomic.
ii) M is Artinian.
iii) M is cyclic and not isomorphic to D.
iv) M is a torsion D-module.
Furthermore, if M is non-zero and satisfies these conditions, then M has finite length $l(M) \leq e(M)$.

PROOF. Clearly, all conditions are satisfied if $M = 0$, as we may assume that $M \neq 0$. First, assume that M is holonomic. Then M has finite length $l(M) \leq e(M)$: If $M_0 \subseteq M_1 \subseteq \dots \subseteq M_n \subseteq \dots$ is a chain a submodules of M, then $e(M_i) < e(M_{i+1})$ for all integers i, since each submodule of M is holonomic. But $e(M)$ is finite, so the length of such chains is bounded by $l(M) \leq e(M)$. In particular, M is Artinian. Secondly, assume that M is Artinian. Since D is a simple ring, but not left Artinian, M is cyclic by Björk [**3**], theorem 1.8.18, and M is not isomorphic to D since D is not left Artinian. Thirdly, assume that M is cyclic, but not isomorphic to D. Then $M \cong D/I$ for some non-zero left ideal $I \subseteq D$, and M is holonomic by lemma 3.12. So the three first conditions in the proposition are equivalent. But M is holonomic if and only if M is a torsion D-module: Assume that M is a torsion D-module, generated by $m_1, \dots, m_n$. Then Dm_i is a cyclic D-module not isomorphic to D for all i, since m_i is a torsion element. So $Dm_i \subseteq M$ is holonomic for all i, and M is a sum of holonomic modules, and therefore holonomic. Conversely, assume that M is holonomic, and let $m \in M$ be a non-zero element. Consider the homomorphism $D \to M$ of D-modules given by $P \mapsto Pm$ for all $P \in D$. Since $Dm \subseteq M$, Dm is holonomic. So if the homomorphism is injective, D is holonomic as well, which is a contradiction. Consequently, $m \in M$ is a torsion element, and M is a torsion D-module. $\square$

Let M be a D-module. If M is non-holonomic, then there exists an element $m \in M$ which is not a torsion element by proposition 3.13. This element gives rise to an exact sequence of D-modules

$$0 \to D \to M \to M/Dm \to 0,$$

where $D \to M$ is defined by right multiplication with m. If M/Dm is non-holonomic, we have $e(M/Dm) = e(M) - 1$. This argument points in the direction of the following structural result:

Proposition 3.14. *Let M be a non-holonomic D-module with $n = e(M)$. Then there exists an injective D-module homomorphism $\phi : D^n \to M$ such that the cokernel of ϕ is an holonomic D-module.*

PROOF. We shall find elements $m_1, \dots, m_n$ such that the corresponding morphism of D-modules $\phi_n : D^n \to M$, given by $\phi_n(P_1, \dots, P_n) = P_1 m_1 + \dots + P_n m_n$, is injective. From the argument preceeding the proposition, we know that we can find an element $m_1 \in M$ such that $\phi_1 : D \to M$ is injective. If $n = 1$, we are done. We show that if $m_1, \dots, m_l \in M$ such that $\phi_l : D^l \to M$ is injective and $1 \leq l < n$, then there exists an element $m_{l+1} \in M$ such that $\phi_{l+1} : D^{l+1} \to M$ is injective: Since $l < n$, $\operatorname{coker} \phi_l$ is non-holonomic, and there exists some $m_{l+1} \in M$ such that the image of m_{l+1} in $\operatorname{coker} \phi$ is not a torsion element. But this means that $Pm_{l+1} \in Dm_1 + \dots + Dm_l$ implies $P = 0$ for all $P \in D$, so the morphism $\phi_{l+1} : D^{l+1} \to M$ is injective. So by induction on l, there exist $m_1, \dots, m_n \in M$ such that $\phi_n : D^n \to M$ is an injection of D-modules. But assume that $\operatorname{coker} \phi_n$ is non-holonomic. Then $e(\operatorname{coker} \phi_n) = e(M) - n = 0$, which is a contradiction. Consequently, $\operatorname{coker} \phi_n$ is holonomic. $\square$

We denote by $\operatorname{Iso}(R)$ the set of isomorphism classes of left R-modules of finite type for any k-algebra R. Let S be a subset of $\operatorname{Iso}(R)$. We define the set of extension of extensions of S in the following way: Let $E_0(S) = S$, and for all integers $n \geq 1$, we define $E_n(S) \subseteq \operatorname{Iso}(R)$ inductively as the set of isomorphism classes of R-modules which are extensions of a R-module in $E_{n-1}(S)$ with an R-module in S. We say that an R-module M is an *extension of extensions* of S if the isomorphism class of M is in $E^n(S)$ for some integer $n \geq 0$, and we denote by $E(S) = \cup E_n(S)$ the set of extensions of extensions of S.

Let F denote the subset of $\mathrm{Iso}(D)$ consisting of the isomorphism classes of the simple D-modules and of the trivial D-module D. We refer to $F \subseteq \mathrm{Iso}(D)$ as the *fundamental set of isomorphism classes* of D-modules. From proposition 3.14, we see that for any D-module M, there is a finite subset $F(M) \subseteq F$ such that M is an extension of extensions of $F(M)$. If M is holonomic, then $F(M)$ consists of simple modules, and the well-known results on composition series of modules of finite length applies. If M is non-holonomic, then $F(M)$ consists of the trivial D-module, and possibly a finite number of simple D-modules.

A simple example will clarify the situation in the non-holonomic case: Let $M = D$ be the trivial D-module. Then, for any non-zero differential operator $P \in D$, we have an exact sequence

$$0 \to D \to D \to D/DP \to 0,$$

where $D \to D$ is given by right multiplication with P. So D can be obtained as an extension of the holonomic D-module D/DP with the trivial D-module D, or it can simply be considered as the trivial D-module D. Furthermore, there is no bound on the length of the holonomic D-module D/DP, when $P \in D$ is non-zero: Indeed, if $\alpha \in k \setminus \mathbf{Z}$, then $D/D(E-\alpha)$ is a simple module by Dixmier [**11**], lemma 24, and $D/D(E-\alpha)^n$ is a holonomic D-module of length n.

We end this section by making some comments on the Morita equivalence: Let Γ be a numerical semigroups, let $A = k[\Gamma]$ be the corresponding monomial curve, and let D be the ring of differential operators on A. Furthermore, let $\overline{D}$ be the first Weyl algebra $A_1(k)$ corresponding to the numerical semigroup $\mathbf{N_0}$. Then, we know from corollary 2.9 that D and $\overline{D}$ are Morita equivalent, so any D-module M corresponds to a uniquely defined $\overline{D}$-module $\overline{M}$. It is clear that M is Artinian if and only if $\overline{M}$ is Artinian, and if this is the case, then $l(M) = l(\overline{M})$. Consequently, the dimension $d(M)$ and the length $l(M)$ are Morita equivalent invariants of D-modules over monomial curves. However, it is not known if the multiplicity $e(M)$ is a Morita equivalent invariant.

4. The characteristic variety

Let M be any D-module, and let (M_n) be a good filtration of M. We denote by $I(M)$ the *characteristic ideal* of M, which is defined by the equation

$$I(M) = \mathrm{rad}(\mathrm{ann}(\mathrm{gr}\, M)).$$

It is clear that $I(M) \subseteq \mathrm{gr}\, D$ is an homogeneous ideal. We shall prove that it is independent upon the chosen good filtration of M.

Lemma 3.15. *Let M be a D-module, let $(M_n), (M'_n)$ be two filtrations of M, and let $I(M), I(M)'$ be the characteristic ideals of M with respect to these filtrations. If $(M_n), (M'_n)$ are good filtrations of M, then $I(M) = I(M)'$.*

Proof. By symmetry, it is enough to show that $I(M) \subseteq I(M)'$, and since these ideals are homogeneous, it is enough to show that any homogeneous element in $I(M)$ is in $I(M)'$. So let $f \in I(M)$ be homogeneous of degree s, represented by $P \in \mathrm{B}^s(A)$. Since $f \in I(M)$, we see that $P^m M_n \subseteq M_{n+ms-1}$ for all integers n. By q iterations of this formula, we see that $P^{qm} M_n \subseteq M_{n+q(ms-1)}$ for all integers n and for all $q \geq 1$. But from lemma 3.8, we know that there exists some integer p such that $M'_{n-p} \subseteq M_n \subseteq M'_{n+p}$ for all integers n, since $(M_n), (M'_n)$ are good filtrations of M. Let $q = 2p+1$, then we have

$$P^{qm} M'_n \subseteq P^{qm} M_{n+p} \subseteq M_{n+p+q(ms-1)} \subseteq M'_{n+2p+q(ms-1)} = M'_{n+qms-1}$$

for all integers n. Consequently, $f^{qm} \in \mathrm{ann}\,\mathrm{gr}\, M$, where $\mathrm{gr}\, M$ is the graded module associated to the filtration (M'_n). This means that $f \in I(M)'$, which is what we wanted to prove. $\square$

We see that if M, M' are isomorphic D-modules, then $I(M) = I(M')$. It follows that the characteristic ideal $I(M)$ is an invariant of M. Furthermore, $I(M) = \operatorname{gr} D$ if and only if $M = 0$.

Let $V(\Gamma)$ be the affine variety $V(\Gamma) = \operatorname{Spec}(\operatorname{gr} D)$ corresponding to the commutative affine semigroup ring $\operatorname{gr} D$. We define the *characteristic variety* $\operatorname{Char}(M)$ of a non-zero D-module M to be the the closed subspace $\operatorname{Char}(M) = V(I(M)) \subseteq V(\Gamma)$. This is an affine scheme over k with corresponding k-algebra $(\operatorname{gr} D)/I(M)$, which is a reduced k-algebra of finite type, but it is not in general an integral domain. It follows that $\operatorname{Char}(M)$ is a (not necessarily irreducible) affine variety. By definition, $(\operatorname{gr} D)/I(M)$ has Krull dimension $d(M)$, so $\operatorname{Char}(M)$ has dimension $d(M)$ as well. It is clear that $\operatorname{Char}(M)$ is an invariant of M.

Proposition 3.16. *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of D-modules. Then* $\operatorname{Char}(M) = \operatorname{Char}(M') \cup \operatorname{Char}(M'')$.

PROOF. Choose a good filtration of M, and let M', M'' have the induced filtrations. Then we obtain an exact sequence of graded $\operatorname{gr} D$-modules associated to these filtrations, $0 \to \operatorname{gr} M' \to \operatorname{gr} M \to \operatorname{gr} M'' \to 0$. From this exact sequence, we see that we have inclusions of homogeneous ideals

$$\operatorname{ann}(\operatorname{gr} M') \operatorname{ann}(\operatorname{gr} M'') \subseteq \operatorname{ann}(\operatorname{gr} M) \subseteq \operatorname{ann}(\operatorname{gr} M') \cap \operatorname{ann}(\operatorname{gr} M'').$$

But $V(ab) = V(a \cap b) = V(a) \cup V(b)$ for arbitrary ideals a, b, and this proves the proposition. □

We see that $\operatorname{Char}(M) = V(\Gamma)$ if M is a non-holonomic D-module, since we have $\operatorname{Char}(D) = V(\Gamma)$, and $D^n \subseteq M$ with $n = e(M) > 0$ for all non-holonomic D-modules M. Furthermore, $\operatorname{Char}(M)$ is a curve for all non-zero D-modules which are holonomic.

Let $P \in D$ be a non-zero differential operator, and consider the holonomic D-module $M = D/DP$, with free resolution given by

$$0 \to D \to D \to M \to 0,$$

where $D \to D$ is given by right multiplication by P. Let the middle term have the Bernstein filtration, which is a good filtration. Then the other terms have induced, good filtrations, and we obtain the following exact sequence of $\operatorname{gr} D$-modules

$$0 \to \operatorname{gr} D \to \operatorname{gr} D \to \operatorname{gr} M \to 0,$$

where the homomorphism $\operatorname{gr} D \to \operatorname{gr} D$ is given by (right) multiplication of the equivalence class $\overline{P} \in \operatorname{gr} D$. It follows that $\operatorname{ann}(\operatorname{gr} M)$ is the homogeneous, principle ideal in $\operatorname{gr} D$ generated by $\overline{P}$, and the characteristic variety is the (not necessarily irreducible) curve $V(\overline{P}) \subseteq V(\Gamma)$.

Let us calculate some examples: Let $\Gamma = \mathbf{N_0}$, then the corresponding ring of differential operators is $D = A_1(k)$. Clearly, $\operatorname{gr} D = k[t, \xi]$ for $\Gamma = \mathbf{N_0}$. We consider the D-module $M = D/DP$: First, let $P = \partial$, then $M \cong A = k[t]$, and $\overline{P} = \xi$. Consequently, $\operatorname{Char}(M) = V(\xi) \subseteq \mathbf{A}^2$, so $\operatorname{Char}(M) \cong \mathbf{A}^1$ in this case. Secondly, let $P = E = t\partial$. Then $\overline{P} = t\xi$, and $\operatorname{Char}(M) = V(t\xi) \subseteq \mathbf{A}^2$, which is a curve with two irreducible components and an isolated singularity in the origin.

It is not hard to see that for any numerical semigroup Γ, the simple D-module $M = A = k[\Gamma]$ has characteristic variety $\operatorname{Char}(M) = \operatorname{Spec} A$. Consequently, the isomorphism class of the characteristic variety is not invariant under the Morita equivalence between D and $\overline{D}$.

5. The Morita equivalence

Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let $D = \mathrm{D}(A)$ be the ring of differential operators on A. Then, we know from corollary 2.9 that D is Morita equivalent to the Weyl algebra $\overline{D} = A_1(k)$. In this section, we shall study this Morita equivalence in more detail.

Let us denote by D-**Mod** the category of D-modules, and by $\overline{\text{D}}$-**Mod** the category of left $\overline{D}$-modules of finite type. We know that the Morita equivalence of D and $\overline{D}$ means that there exists an equivalence between the category of left D-modules and left $\overline{D}$-modules. Furthermore, a left D-module M is of finite type over D if and only if the corresponding left $\overline{D}$-module $\overline{M}$ is of finite type over $\overline{D}$. So we have equivalences of categories

$$F : \overline{\mathrm{D}}\text{-}\mathbf{Mod} \to \mathrm{D}\text{-}\mathbf{Mod},$$

$$G : \mathrm{D}\text{-}\mathbf{Mod} \to \overline{\mathrm{D}}\text{-}\mathbf{Mod}\,.$$

Since $A \subseteq \overline{A} \subseteq T = k[t, t^{-1}]$, where T may be considered as a localization of A or $\overline{A}$ with respect to a multiplicatively closed subset consisting of homogeneous elements, we may consider the rings $D, \overline{D}$ as graded subrings of $\mathrm{D}(T)$. In fact, we have the identifications of graded rings $D = \{P \in \mathrm{D}(T) : P * A \subseteq A\}$ and $\overline{D} = \{P \in \mathrm{D}(T) : P * \overline{A} \subseteq \overline{A}\}$. We define $\mathrm{D}(\overline{A}, A) = \{P \in \mathrm{D}(T) : P * \overline{A} \subseteq A\}$, and $\mathrm{D}(A, \overline{A}) = \{P \in \mathrm{D}(T) : P * A \subseteq \overline{A}\}$ using these identifications. Clearly, $\mathrm{D}(\overline{A}, A)$ is a D-$\overline{D}$ bimodule in $\mathrm{D}(T)$, and $\mathrm{D}(A, \overline{A})$ is a $\overline{D}$-D bimodule in $\mathrm{D}(T)$.

Lemma 3.17. *We have* $\mathrm{D}(\overline{A}, A) = \mathrm{D}_A(\overline{A}, A)$ *and* $\mathrm{D}(A, \overline{A}) = \mathrm{D}_A(A, \overline{A})$.

Proof. From Smith and Stafford [**32**], lemma 2.7, it follows that the equation $\mathrm{D}_A(\overline{A}, A) = \{P \in \overline{D} : P * \overline{A} \subseteq A\}$ holds. But $\{P \in \overline{D} : P * \overline{A} \subseteq A\} = \mathrm{D}(\overline{A}, A)$ from corollary 1.6, since T is a localization of $\overline{A}$. From Smith and Stafford [**32**], proposition 3.14, we also see that $\mathrm{D}_A(A, \overline{A}) = \{P \in \mathrm{D}(K) : P * A \subseteq \overline{A}\}$, where K is the field of fractions of A. But we have inclusions

$$\mathrm{D}_A(A, \overline{A}) \subseteq \{P \in \mathrm{D}(T) : P * A \subseteq \overline{A}\} \subseteq \{P \in \mathrm{D}(K) : P * A \subseteq \overline{A}\},$$

since $S^{-1}\,\mathrm{D}_A(A, \overline{A}) \cong \mathrm{D}(T)$ when $S = \{x \in A : x \neq 0,\ \text{x is homogeneous }\}$ from Smith and Stafford [**32**], lemma 2.7. Consequently, both inclusions are equalities, and $\mathrm{D}_A(A, \overline{A}) = \mathrm{D}(A, \overline{A})$. □

The results in Smith and Stafford [**32**], section 3.14, show that the equivalences of categories $F : \overline{\mathrm{D}}\text{-}\mathbf{Mod} \to \mathrm{D}\text{-}\mathbf{Mod}$ and $G : \mathrm{D}\text{-}\mathbf{Mod} \to \overline{\mathrm{D}}\text{-}\mathbf{Mod}$ are implemented in the following way: For all $\overline{D}$-modules $\overline{M}$, we have $F(\overline{M}) = P \otimes_{\overline{D}} \overline{M}$, and for all D-modules M, we have $G(M) = P^* \otimes_D M$. Furthermore, P, P^* are the bimodules given by $P = \mathrm{D}(\overline{A}, A)$ and $P^* = \mathrm{D}(A, \overline{A})$.

We know that $\mathrm{D}(T)$ is $\mathbf{Z}$-graded k-algebra with graded subrings $D, \overline{D}$. Therefore, $\mathrm{D}(T)$ has a natural $\mathbf{Z}$-graded structure as left and right D-module and as left and right $\overline{D}$-module. Let $P \in \mathrm{D}(T)$ be a non-zero differential operator, and let $P = \sum P_w$ be the unique decomposition in homogeneous differential operators P_w of weight w. If $P \in \mathrm{D}(\overline{A}, A)$, then $P_w \in \mathrm{D}(\overline{A}, A)$ for all integers w, and if $P \in \mathrm{D}(A, \overline{A})$, then $P_w \in \mathrm{D}(A, \overline{A})$ for all integers w: This is clear, since A and $\overline{A}$ are $\mathbf{Z}$-graded rings. Consequently, $\mathrm{D}(\overline{A}, A)$ and $\mathrm{D}(A, \overline{A})$ are $\mathbf{Z}$-graded sub-bimodules of $\mathrm{D}(T)$.

We shall in fact describe the graded structure of these bimodules explicitly, following the strategy used in the proof of theorem 2.2: For any pair Γ, Γ' of numerical semigroups, we write $\Omega_w(\Gamma/\Gamma') = \{\gamma \in \Gamma : \gamma + w \notin \Gamma'\}$ for all integers $w \in \mathbf{Z}$. Then clearly, $\Omega_w(\Gamma/\Gamma')$ is a finite set, and we define the *characteristic polynomial* of w relative to Γ/Γ' to be the polynomial in $k[\xi]$ given by

$$\chi_w(\Gamma/\Gamma') = \prod_{\gamma \in \Omega_w(\Gamma/\Gamma')} (\xi - \gamma)$$

for all integers w. Clearly, this is a monic polynomial, and its degree equals the cardinality of $\Omega_w(\Gamma/\Gamma')$. Using this notation, we obtain:

Proposition 3.18. *Let Γ be any numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let D be the ring of differential operators of A. Then we have*

$$\mathrm{D}(\overline{A}, A) = \bigoplus_w P_w^- \; k[E],$$
$$\mathrm{D}(A, \overline{A}) = \bigoplus_w P_w^+ \; k[E],$$

where $P_w^-, P_w^+ \in \mathrm{D}(T)$ are differential operators given by $P_w^- = t^w \chi_w(\mathbf{N_0}/\Gamma)(E)$ and $P_w^+ = t^w \chi_w(\Gamma/\mathbf{N_0})(E)$ for all integers w.

Proof. This follows from the proof of theorem 2.2 and corollary 2.3. □

An important consequence of these results, is that the Morita equivalence induces an equivalence between $\mathbf{Z}$-graded D-modules and $\mathbf{Z}$-graded $\overline{D}$-modules:

Proposition 3.19. *Let $\overline{M}$ be a $\mathbf{Z}$-graded $\overline{D}$-module, and let M be a $\mathbf{Z}$-graded D-module. Then $F(\overline{M})$ is a $\mathbf{Z}$-graded D-module and $G(M)$ is a $\mathbf{Z}$-graded $\overline{D}$-module in a natural way.*

Proof. This is clear, since a tensor product of two graded modules has a natural graded structure, see Năstăsescu and Van Oystaeyen [**29**], section I.3.4. □

6. Localizations

Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let $D = \mathrm{D}(A)$ be the ring of differential operators on A. Since A is an integral domain, the set $S = A \setminus \{0\}$ is a multiplicatively closed subset of A. We shall denote by $K = S^{-1}A$ the *field of fractions* of A, and we see that $K = k(t)$ for all numerical semigroups Γ. Clearly, the localization map $A \to K$ is injective, and we shall always identify the monomial curve $A = k[\Gamma]$ with its image in K, and also identify $\overline{A}$ with its image in K via $\overline{A} = k[\mathbf{N_0}]$.

From proposition 1.5, we see that the non-commutative ring D has a localization with respect to any multiplicatively closed subset in A. In particular, this applies to the set S, so S is an *Ore set* for D. We also see that $\mathrm{D}(K)$ is the localization of D with respect to S. Equivalently, there is a natural isomorphism of K-D bimodules $K \otimes_A D \cong \mathrm{D}(K)$.

Let us give an explicit description of the ring $\mathrm{D}(K)$: We observe that in particular, we have an isomorphism $K \otimes_A D \cong \mathrm{D}(K)$ for the numerical semigroup $\Gamma = \mathbf{N_0}$, so $\mathrm{D}(K) \cong k(t) \otimes_{k[t]} \overline{D}$, where $\overline{D} = A_1(k)$ is the Weyl algebra. This means that any non-zero differential operator $P \in \mathrm{D}(K)$ can be written uniquely in the form

$$P = \sum_{i=0}^{p} c_i \partial^i,$$

where $c_i \in k(t)$ is a rational function for $0 \leq i \leq p$, and $p = d(P)$ is the order of P.

Lemma 3.20. *The ring $\mathrm{D}(K)$ has a left and right Euclidean division algorithm. In particular, $\mathrm{D}(K)$ is a PI domain.*

Proof. Let P, Q be non-zero differential operators in $\mathrm{D}(K)$ of orders p and q. We show by induction on p that there exist differential operators L, R such that $P = LQ + R$ with $d(R) < q$: If $p = 0$, then we let $L = 0, R = P$ if $q > 0$, and we let $L = P/Q$ and $R = 0$ if $q = 0$. Let us assume that $p > 0$. If $p < q$, then we let $L = 0$, $R = P$, and there is nothing to prove. If $p \geq q$, write $c_p \partial^p$ and $c'_q \partial^q$

for the leading coefficients of P and Q, and write $P' = P - (c_p/c'_q \partial^{p-q})Q$. Then clearly, $d(P') < d(P)$, so by the induction hypothesis, $P' = L'Q + R$ for differential operators $L', R \in \mathrm{D}(K)$ with $d(R) < q$. We obtain $P = (L' + c_p/c'_q \partial^{p-q})Q + R'$. So by induction, $\mathrm{D}(K)$ has a left Euclidean division algorithm, and by a similar argument it has a right Euclidean division algorithm, as well. $\square$

A direct computation shows that $d(PQ) = d(P) + d(Q)$ for all differential operators $P, Q \in \mathrm{D}(K)$. This means that all units in the ring $\mathrm{D}(K)$ have order 0, and since K is a field, the units are exactly $\mathrm{D}(K)^* = K^* = K \setminus \{0\}$.

Let M be a D-module. We say that an element $m \in M$ is an *S-torsion element* if $sm = 0$ for some $s \in S$. That is, m is an S-torsion element if and only if $\mathrm{ann}_A(m) = \mathrm{ann}_D(m) \cap A \subseteq A$ is a non-zero ideal, or equivalently, if the element $m \in M$ is a torsion element when M is considered as an A-module. We denote by

$$T_S(M) = \{m \in M : sm = 0 \text{ for some } s \in S\} \subseteq M$$

the set of S-torsion elements in M. This is a D-module, since S is an Ore set for D. We say that M is *S-torsion free* if $T_S(M) = 0$, and that M is an *S-torsion module* if $T_S(M) = M$. Notice that if M is a simple D-module, then M is an S-torsion free module or an S-torsion module.

Let M be any left $\mathrm{D}(K)$-module of finite type. We define the *degree* of N to be $\deg N = \dim_K N$. Clearly, the degree is an invariant of N, but it is not finite in general. However, if N is a cyclic $\mathrm{D}(K)$-module which is not isomorphic to $\mathrm{D}(K)$, then $\deg N$ is finite: Let $n \in N$ be a cyclic generator, and put $I = \mathrm{ann}(n) \subseteq \mathrm{D}(K)$. Then I is a non-zero left ideal in $\mathrm{D}(K)$, and therefore generated by some non-zero differential operator $P \in \mathrm{D}(K)$. Let $d(P) = d$, then we may choose P with leading term ∂^p, since K^* are units in $\mathrm{D}(K)$. So the set $\{n, \partial n, \dots, \partial^{d-1} n\}$ is a basis for N as a K-linear space, and $\deg N = d$.

Let M be a D-module, then clearly $S^{-1}M = S^{-1}D \otimes_D M$ is a left $\mathrm{D}(K)$-module of finite type. We define the degree of M to be $\deg M = \deg S^{-1}M$, which may be infinite or a non-negative integer, as the case may be. If M is holonomic, then $N = S^{-1}M$ is a cyclic B-module, not isomorphic to B, so $\deg M$ is finite. If M is non-holonomic, then $D^n \subseteq M$ with $n = e(M) \geq 1$ from proposition 3.14. But $\mathrm{D}(K)$ has K-linear basis $\{\partial^i : i \geq 0\}$, so $\deg \mathrm{D}(K)$ is infinite. Consequently, $\deg M$ is infinite for any non-holonomic D-module M.

Proposition 3.21. *Let M be a D-module. Then the degree of M is finite if and only if M is holonomic, and the degree is an additive function on the category of holonomic D-modules. Furthermore, we have $\deg M = 0$ if and only if M is an S-torsion module.*

There is a natural covariant functor $S^{-1} : \mathrm{D}\text{-}\mathbf{Mod} \to \mathrm{D(K)}\text{-}\mathbf{Mod}$, given by $M \mapsto S^{-1}M$ for all D-modules M. Since it is a localization functor, it follows that S^{-1} is an exact functor. Moreover, we have a diagram of functors

$$\begin{tikzcd}
 & \mathrm{D(K)}\text{-}\mathbf{Mod} & \\
\mathrm{D}\text{-}\mathbf{Mod} \arrow[ur, "S^{-1}"] \arrow[rr, shift left, "G"] & & \overline{\mathrm{D}}\text{-}\mathbf{Mod} \arrow[ul, "(\overline{S})^{-1}"'] \arrow[ll, shift left, "F"]
\end{tikzcd}$$

where $\overline{S} = \overline{A} \setminus \{0\}$. We show that this is a commutative diagram of functors, in the sense that the localization functors commute with the equivalence of categories between D-$\mathbf{Mod}$ and $\overline{\mathrm{D}}$-$\mathbf{Mod}$:

Proposition 3.22. *Let M be a D-module, and let $\overline{M} = G(M)$ be the $\overline{D}$-module corresponding to M under the Morita equivalence. Then $S^{-1}M \cong (\overline{S})^{-1}\overline{M}$. In particular, the degree of a D-module is invariant under Morita equivalence.*

Proof. Clearly, $S \subseteq \overline{S} \subseteq \overline{A}$ is a multiplicatively closed subset of $\overline{A}$. Since $(\overline{S})^{-1}\overline{M} \cong K \otimes_{\overline{A}} \overline{M} \cong S^{-1}\overline{M}$, it is enough to prove $S^{-1}M \cong S^{-1}\overline{M}$. But $\overline{M} = \mathrm{D}(A, \overline{A}) \otimes_D M$, and $S^{-1}\,\mathrm{D}(A, \overline{A}) \cong \mathrm{D}(K)$ from Smith and Stafford [**32**], remark 1.3 (d). So we have $S^{-1}\overline{M} \cong \mathrm{D}(K) \otimes_D M \cong K \otimes_A M \cong S^{-1}M$. $\square$

The following observation will be useful later on: From proposition 3.21, we see that a D-module is an S-torsion module if and only if $\deg M = 0$. Furthermore, proposition 3.22 shows that the degree of a module is invariant under Morita equivalence. So it follows that the property of being an S-torsion module is invariant under Morita equivalence.

7. Graded localizations

We shall adapt the localization techniques described in section 6, which were introduced in Block [**4**], to the graded situation. So for the rest of this section, we let S be the multiplicatively closed subset $S = \{x \in A : x \neq 0, x \text{ is homogeneous}\}$. Then $S^{-1}A \cong T$ is a graded ring, and $S^{-1}D \cong \mathrm{D}(T) \cong T \otimes_A D$ is graded as well.

We say that a D-module M is a *graded D-module* if M has a $\mathbf{Z}$-graded structure $M = \oplus M_i$ compatible with the natural graded structure on the k-algebra D. For any graded module M and any integer n, we denote by $M[n]$ the *n'th twisted D-module* of M. $M[n]$ is a graded D-module with the same underlying D-module structure as M, such that $M[n]_i = M_{n+i}$ for all integers i. We see that any graded D-module M has a localization $S^{-1}M = \mathrm{D}(T) \otimes_D M$ which is a graded $\mathrm{D}(T)$-module.

We already know the ring structure of $\mathrm{D}(T)$ quite well. We recall that any homogeneous differential operator $P \in \mathrm{D}(T)$ of weight w and order p has the form

$$P = \sum_{i=0}^{p} c_i t^{i+w} \partial^i$$

with $c_i \in k$ for $0 \leq i \leq p$ and $c_p \neq 0$. Furthermore, the graded left and right ideals in $\mathrm{D}(T)$ are principal:

Lemma 3.23. *Let P, Q be homogeneous differential operators in $\mathrm{D}(T)$ with $Q \neq 0$. Then there exist unique homogeneous differential operators L, R and L', R' in $\mathrm{D}(T)$ such that $P = LQ + R$, $P = QL' + R'$ and $d(R), d(R') < d(Q)$. In particular, any left or right graded ideal in $\mathrm{D}(T)$ is principal.*

Proof. We may assume that $P \neq 0$, so let $p = d(P), q = d(Q)$ and let w, w' be the weights of P, Q. We show that there exist homogeneous differential operators L, R of weights $w - w', w$ such that $P = LQ + R$ and $d(R) < q$ (where we allow L, R to be zero) by induction on p: If $p = 0$, then we let $L = 0, R = P$ if $q > 0$, and we let $L = PQ^{-1}$ and $R = 0$ if $q = 0$. Let us assume that $p > 0$. If $p < q$, then we let $L = 0$, $R = P$, and there is nothing to prove. If $p \geq q$, write $c_p t^{p+w}\partial^p$ and $c'_q t^{q+w'}\partial^q$ for the leading coefficients of P and Q, and write $P' = P - (c_p/c'_q t^{p+w-q-w'}\partial^{p-q})Q$. Then clearly, P' is a homogeneous differential operator of weight w, and we have $d(P') < d(P)$, so by the induction hypothesis, $P' = L'Q + R$ for homogeneous differential operators $L', R \in \mathrm{D}(T)$ of weights $w - w', w$ which satisfy $d(R) < q$. We obtain $P = (L' + c_p/c'_q t^{p+w-q-w'}\partial^{p-q})Q + R$. So the existence of L, R follows by induction, and the uniqueness is clear. The right division algorithm can be proved in a similar way. $\square$

Let M be a graded D-module. We say that an element $m \in M$ is an *S-torsion element* if $sm = 0$ for some $s \in S$, and we write $T_S(M) \subseteq M$ for the subset of M consisting of S-torsion elements. Let $m = \sum m_i$ be the unique decomposition of $m \in M$ in homogeneous elements $m_i \in M_i$. Then $m \in T_S(M)$ if and only if there exists a homogeneous element $s \in S$ such that $sm_i = 0$ for all i. Since S is an Ore set with respect to D, $T_S(M)$ is a invariant under D, and hence $T_S(M) \subseteq M$ is a graded D-submodule. We say that M is an *S-torsion module* if $T_S(M) = M$, and that M is an *S-torsion free module* if $T_S(M) = 0$.

For any $\mathbf{Z}$-graded ring R, we say that a graded R-module M is gr-simple if $0, M$ are the only homogeneous R-submodules of M. Clearly, every graded R-module which is simple, is gr-simple. The following proposition shows that the converse statement holds for graded D-modules. We shall therefore make no distinction between gr-simple D-modules and simple, graded D-modules in the remainder of this thesis.

Proposition 3.24. *Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let D be the ring of differential operators on A. Then a graded D-module M is gr-simple if and only if it is simple.*

PROOF. Let M be a gr-simple D-module M, and let $m \in M$ be a non-zero, homogeneous element. We denote by $I = \{P \in D : Pm = 0\}$ the annihilator of m. There is an isomorphism of graded D-modules $D/I \to M$ of degree $\deg(m)$, given by right multiplication with m. Since $I \neq D$, M is an Artinian D-module by lemma 3.12, and in particular M is not 1-critical. But from Năstăsescu and Van Oystaeyen [**29**], theorem II.7.5, any gr-simple D-module M has an underlying D-module structure which is either simple or 1-critical. So it follows that the D-module M is simple, and this proves the proposition. □

There is a natural functor $S^{-1} : \text{D-}\mathbf{Mod} \to \text{D(T)-}\mathbf{Mod}$, given by $M \mapsto S^{-1}M$, and we have seen that if M is a graded D-module, then $S^{-1}M$ is a graded D(T)-module. By an argument very similar to the proof of proposition 3.22, we have the following diagram of functors, where all the triangles commute:

$$\begin{tikzcd} & \text{D(T)-}\mathbf{Mod} & \\ \text{D-}\mathbf{Mod} \arrow[ur, "S^{-1}"] \arrow[rr, shift left, "G"] & & \overline{\text{D}}\text{-}\mathbf{Mod} \arrow[ul, "(\overline{S})^{-1}"'] \arrow[ll, shift left, "F"] \end{tikzcd}$$

Furthermore, all these functors preserve the relevant graded structures. We also remark that if M is a graded D-module, then $S^{-1}M = 0$ if and only if M is an S-torsion module.

For any associative k-algebra R, let us denote by Simple(R) the set of isomorphism classes of simple left R-modules and by Simple(R)[P] $\subseteq$ Simple(R) the subset of isomorphism classes which satisfy the property P. Then we have a canonical, injective map S^{-1} : Simple(D)[$S-$ torsion free] $\to$ Simple(D(T)) by Block [**4**], lemma 2.2.1. Since Block [**4**], corollary 2.2.2 holds for the ring D, the above map is also surjective by an argument similar to that of Block [**4**], corollary 2.2. We conclude that S^{-1} : Simple(D)[$S-$ torsion free] $\to$ Simple(D(T)) is a bijective map.

For any $\mathbf{Z}$-graded associative k-algebra R, let us denote by gr-Simple(R) the set of equivalence classes of simple, graded R-modules, where the equivalence relation is given by graded isomorphisms of degree 0 and twists. Furthermore, we denote by gr-Simple(R)[P] $\subseteq$ gr-Simple(R) the subset of equivalence classes that satisfy the property P. We have already seen that if $R = D$, then we may replace the condition graded and simple with the condition gr-simple. Since $\mathrm{D}(T)$ is a PI ring, the same holds for $R = \mathrm{D}(T)$.

Lemma 3.25. *The natural map* $\text{gr-Simple}(D) \to \text{Simple}(D)$ *is injective.*

PROOF. Assume that $I, J \subseteq D$ are maximal, homogeneous left ideals in D, so that $M = D/I$ and $N = D/J$ are simple, graded D-modules. Assume furthermore that $\phi : M \to N$ is an isomorphism of D-modules. Then ϕ is given by right multiplication with P for some $P \in D$. Let $P = P_1 + \cdots + P_n$, where P_i is homogeneous and non-zero for $1 \leq i \leq n$. We know that $IP \subseteq J$, so $IP_i \subseteq J$ for all i, since I, J are homogeneous ideals. This means that we have well-defined homogeneous morphisms $\phi_i : M \to N$ for all i, where ϕ_i is given by right multiplication with P_i. Since M, N are simple modules, we have $\phi_i = 0$ or ϕ_i is an isomorphism for all i. Since $\phi = \phi_1 + \cdots + \phi_n$ is an isomorphism, it follows that there exists an index i such that ϕ_i is an isomorphism. So $\phi_i : M \to N$ is a graded isomorphism. $\square$

Observe that if M is a simple, graded D-module, then either M is an S-torsion module and $S^{-1}M = 0$, or M is an S-torsion free module and $S^{-1}M$ is a simple $\mathrm{D}(T)$-module: The last part follows from Block [**4**], lemma 2.1. So we have the following commutative diagram of sets:

$$\begin{tikzcd}
\text{Simple}(D)[S - \text{torsion free}] \arrow[r] & \text{Simple}(\mathrm{D}(T)) \\
\text{gr-Simple}(D)[S - \text{torsion free}] \arrow[r] \arrow[u] & \text{gr-Simple}(\mathrm{D}(T)) \arrow[u]
\end{tikzcd}$$

We know that the upper horizontal map is a bijection, and that the left vertical map is an injection. So it is clear that the lower horizontal map is injective. To see that it is surjective, assume that N is a simple, graded $\mathrm{D}(T)$-module, and let $n \in N$ be a non-zero, homogeneous element. Then there is an isomorphism of graded modules $D/J \to N$ of degree $\deg(n)$, where J is the ideal $J = \{P \in \mathrm{D}(T) : Pn = 0\} \neq 0$. Let $I = J \cap D$, then $I \subseteq D$ is a non-zero, homogeneous left ideal. So $D/I \cong Dn \subseteq N$ is a graded D-submodule of N which is Artinian. Then we may find a simple, graded D-module $M \subseteq Dn \subseteq N$: Since the graded D-submodules of Dn satisfy the ACC and DCC, we can find a gr-simple submodule $M \subseteq Dn$, and any gr-simple D-module is a simple, graded D-module by proposition 3.24. So Block [**4**], lemma 2.2.1 shows that $S^{-1}M = N$. This proves the following proposition:

Proposition 3.26. *Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let D be the ring of differential operators on A. If $S = \{x \in A : x \neq 0, x \text{ is homogeneous }\}$, the map*

$$S^{-1} : \text{gr-Simple}(D)[S - \textit{torsion free}] \to \text{gr-Simple}(\mathrm{D}(T))$$

is a bijection.

Let $P \in \mathrm{D}(T)$ be a non-zero, irreducible differential operator, which is homogeneous of weight w. Then we have $P = t^w f(E)$, where t^w is a unit in $\mathrm{D}(T)$ and $f(E)$ is a polynomial in E with coefficients in k such that $\deg f = 1$. So any maximal homogeneous ideal I in $\mathrm{D}(T)$ has the form $I = \mathrm{D}(T)P$ with $P = E - \alpha$ for some $\alpha \in k$.

If $\alpha = 0$, then $I = \mathrm{D}(T)\partial$ and $M = \mathrm{D}(T)/I$ is a simple, graded $\mathrm{D}(T)$-module. Moreover, M is isomorphic to the left $\mathrm{D}(T)$-module T, with $\mathrm{D}(T)$-module structure given by $Pf(t) = P * f(t)$ for all $f(t) \in T$. If $\alpha \in \mathbf{Z}$, consider the exact sequence of left $\mathrm{D}(T)$-modules

$$0 \to \mathrm{D}(T)(E - \alpha) \to \mathrm{D}(T) \to T \to 0,$$

where the morphism $\mathrm{D}(T) \to T$ is given by $P \mapsto Pt^\alpha = P * t^\alpha$ for all differential operators $P \in \mathrm{D}(T)$. Since this morphism is homogeneous of degree α, we see that the simple, graded left $\mathrm{D}(T)$-module $\mathrm{D}(T)/\,\mathrm{D}(T)(E-\alpha)$ is isomorphic to $T[\alpha]$ as a graded $\mathrm{D}(T)$-module.

Let N_α be the simple, graded $\mathrm{D}(T)$-module $N_\alpha = \mathrm{D}(T)/\,\mathrm{D}(T)(E-\alpha)$ for all $\alpha \in k$. The remarks above show that any simple, graded $\mathrm{D}(T)$-module in $\text{gr-Simple}(\mathrm{D}(T))$ is equivalent to N_α for some $\alpha \in k$. Furthermore, we have shown N_α is equivalent with N_0 for all $\alpha \in \mathbf{Z}$. More generally, we have:

Lemma 3.27. *Let $\alpha, \beta \in k$. The simple, graded $\mathrm{D}(T)$-modules N_α, N_β are equivalent in $\text{gr-Simple}(\mathrm{D}(T))$ if and only if $\alpha - \beta \in \mathbf{Z}$.*

Proof. If $\alpha - \beta = n \in \mathbf{Z}$, then right multiplication by t^n induces an isomorphism of graded modules $N_\alpha \to N_\beta$ of degree n. Assume that $n = \alpha - \beta \notin \mathbf{Z}$, and denote by $D = \mathrm{A}_1(k)$ the first Weyl algebra and by M_γ the graded D-module $M_\gamma = D/D(E-\gamma)$ for all $\gamma \in k \setminus \mathbf{Z}$. If $\alpha, \beta \notin \mathbf{Z}$, it follows from Dixmier [**11**], lemma 24 that M_α, M_β are non-isomorphic simple modules. If $\alpha \in \mathbf{Z}$ or $\beta \in \mathbf{Z}$, we may assume that $\alpha \notin \mathbf{Z}$ and $\beta = 0$ by symmetry and the remark preceding the lemma. In this case, M_α and $M_0 = A = k[t]$ are non-isomorphic by Dixmier [**12**], proposition 4.4, and they are simple modules by corollary 2.9 and Dixmier [**11**], lemma 24. So in all cases, M_α and M_β define different equivalence classes in $\text{gr-Simple}(\overline{D})[S-\text{torsion free}]$. By proposition 3.26, we see that N_α and N_β are not equivalent in $\text{gr-Simple}(\mathrm{D}(T))$. □

Let $I = I(k) \subseteq k$ be a fixed subset of k containing 0, such that the natural map $I \to k/\mathbf{Z}$ is a bijection of sets. If $k = \mathbf{C}$ is the complex numbers, we can for instance choose $I = \{x + iy \in \mathbf{C} : 0 \leq x < 1\}$. In this notation, we have the following result:

Corollary 3.28. *There is a bijective correspondence between the set $\{N_\alpha : \alpha \in I\}$ and the set $\text{gr-Simple}(\mathrm{D}(T))$ of equivalence classes of simple, graded $\mathrm{D}(T)$-modules.*

CHAPTER 4

Modules with integrable connections

Let A be a commutative k-algebra of finite type, and let M be an A-module. If A is a regular ring, the notions of connections on M and of covariant derivatives on M coincide. We extend both these notions to the general case, where they are different in an essential way. We develop the obstruction theory for connections and for covariant derivatives on M, which is due to Laudal in the general case. Moreover, we present an application to torsion free, graded modules of rank 1 over monomial curves.

In contrast, we mention a result obtained by Henrik Vosegaard and the author, given in theorem 4.17. It shows that existence of D-module structures on M compatible with the A-module structure is a much stronger condition than existence of integrable covariant derivatives on M.

1. Basic definitions

Let A be a commutative k-algebra. We define the *module of Kähler differentials* $\Omega^1(A/k)$ on A/k to be the kernel of the natural A-linear map $\mathcal{P}^1(A/k) \to \mathcal{P}^0(A/k)$. Clearly, $\Omega^1(A/k)$ inherits an A-module structure from $\mathcal{P}^1(A/k)$: This is the A-module structure induced by $j_2 : A \to A \otimes_k A$, see section 1.2. Moreover, there is a derivation $d : A \to \Omega^1(A/k)$, induced by the map $j_1 - j_2 : A \to A \otimes_k A$. This is called the *universal derivation* of A/k. It is easy to see that any element $w \in \Omega^1(A/k)$ has the form $w = \sum a_i d(b_i)$, with $a_i, b_i \in A$ for all i. This implies that the functor $\mathrm{Der}_k(A, -) : \mathrm{Mod}_A \to \mathrm{Mod}_A$ is represented by the couple $(\Omega^1(A/k), d)$, in the sense that the derivation d induces an isomorphism of functors between $\mathrm{Hom}_A(\Omega^1(A/k), -)$ and $\mathrm{Der}_k(A, -)$.

Let $\Omega^n(A/k) = \wedge^n_A \Omega^1(A/k)$ for all $n \geq 0$. This is an A-module for all n, and $\Omega^0(A/k) = A$. There is a k-linear map $d^n : \Omega^n(A/k) \to \Omega^{n+1}(A/k)$ for all $n \geq 0$, such that $d^0 = d$: Let $w \in \Omega^n(A/k)$, then w has the form $w = \sum a_i (d\gamma_{i1} \wedge \cdots \wedge d\gamma_{in})$ with $a_i, \gamma_{ij} \in A$ for all integers i, j, and we define $d^n(w) = \sum da_i \wedge d\gamma_{i1} \wedge \cdots \wedge d\gamma_{in}$. We see that the resulting sequence of k-linear maps

$$A \to \Omega^1(A/k) \to \cdots \to \Omega^n(A/k) \to \dots$$

is a complex of k-vector spaces, called the *algebraic de Rham complex* of A/k.

Let M be any A-module. We define a *connection* on M to be a k-linear map $\nabla : M \to \Omega^1(A/k) \otimes_A M$ which has the property that

$$\nabla(am) = a\nabla(m) + d(a) \otimes m$$

for all $a \in A$, $m \in M$. This property is called the derivation property of the connection ∇.

Assume that ∇ is a connection on M. Then ∇ induces a sequence of k-linear maps

$$M \to \Omega^1(A/k) \otimes_A M \to \Omega^2(A/k) \otimes_A M \to \cdots \to \Omega^n(A/k) \otimes_A M \to \dots$$

where the map $\nabla^n : \Omega^n(A/k) \otimes_A M \to \Omega^{n+1}(A/k) \otimes_A M$ is given by the formula $\nabla^n(w \otimes m) = d^n(w) \otimes m + (-1)^n w \wedge \nabla(m)$ for all $w \in \Omega^n(A/k)$, $m \in M$. We see that $\nabla^0 = \nabla$. Let us denote by $R_\nabla : M \to \Omega^2(A/k) \otimes_A M$ the composition $\nabla^1 \circ \nabla$.

This is a A-linear map, and it is called the *curvature* of the connection ∇. We see that the sequence of k-linear maps is a complex of k-vector spaces if and only if $R_\nabla = 0$. We say that the connection ∇ is *integrable* if $R_\nabla = 0$, and we refer to the above complex as the *de Rham complex* of the integrable connection ∇ on M in this case. It is also usual to call a connection ∇ flat or regular if $R_\nabla = 0$.

Let A be a commutative k-algebra, and let $\mathbf{g}$ be a Lie algebra over k. We say that $(A, \mathbf{g})$ is a *Lie-Cartan pair* over k if there is an action of A on $\mathbf{g}$ and an action of $\mathbf{g}$ on A with the following properties:

i) The action of A on $\mathbf{g}$ makes $\mathbf{g}$ an A-module.
ii) The action of $\mathbf{g}$ on A induces a homomorphism $\rho : \mathbf{g} \to \mathrm{Der}_k(A)$ of k-Lie algebras and of A-modules.
iii) We have $[g, ah] = a[g, h] + \rho_g(a)h$ for all $g, h \in \mathbf{g},\ a \in A$.

We shall often write $\mathbf{g}$ for the Lie-Cartan pair $(A, \mathbf{g})$ when the k-algebra A is understood from the context.

The first example of a Lie-Cartan pair $(A, \mathbf{g})$ is given by $\mathbf{g} = \mathrm{Der}_k(A)$ for any commutative k-algebra A. In this case, the A-module structure on $\mathbf{g}$ and the homomorphism $\mathbf{g} \to \mathrm{Der}_k(A)$ are the natural ones. For another example, consider any k-linear subspace $\mathbf{g} \subseteq \mathrm{Der}_k(A)$, such that $\mathbf{g}$ is an A-submodule and a k-Lie subalgebra of $\mathrm{Der}_k(A)$. In fact, every Lie-Cartan pair $(A, \mathbf{g})$ over k such that $\rho : \mathbf{g} \to \mathrm{Der}_k(A)$ is injective is isomorphic to a Lie-Cartan pair of this type. We shall identify $\mathbf{g}$ with its image in $\mathrm{Der}_k(A)$ when ρ is injective.

Let $(A, \mathbf{g})$ be a Lie-Cartan pair over k, and let M be an A-module. A $\mathbf{g}$-*connection* on M is an A-linear homomorphism $\nabla : \mathbf{g} \to \mathrm{End}_k(M)$ which has the property

$$\nabla_g(am) = a\nabla_g(m) + \rho_g(a)m$$

for all $a \in A,\ m \in M,\ g \in \mathbf{g}$. This property is called the derivation property of the $\mathbf{g}$-connection ∇. If $\mathbf{g} = \mathrm{Der}_k(A)$, we call the $\mathbf{g}$-connection ∇ a *covariant derivative* on M.

Let ∇ be a $\mathbf{g}$-connection on M. Then there exists an A-linear homomorphism $R_\nabla : \mathbf{g} \wedge_A \mathbf{g} \to \mathrm{End}_A(M)$, given by $R_\nabla(g \wedge h) = \nabla_{[g,h]} - [\nabla_g, \nabla_h]$ for all $g, h \in \mathbf{g}$. This homomorphism is called the *curvature* of the $\mathbf{g}$-connection ∇, and we see that $R_\nabla = 0$ if and only if ∇ is a homomorphism of Lie algebras. We say that ∇ is an *integrable* $\mathbf{g}$-connection if $R_\nabla = 0$.

2. Categories of modules with connection

Let A be a commutative k-algebra. We define the *category of modules with integrable connection* $\mathbf{MC}(A)$ in the following way: An object in the category $\mathbf{MC}(A)$ is a couple (M, ∇), where M is an A-module and ∇ is an integrable connection on M. Given a pair of objects $(M, \nabla),\ (M', \nabla')$ in the category $\mathbf{MC}(A)$, we define a morphism from (M, ∇) to (M', ∇') to be a homomorphism $\phi : M \to M'$ of A-modules, such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\nabla"] \arrow[d, "\phi"'] & \Omega^1(A/k) \otimes_A M \arrow[d, "\mathrm{id} \otimes \phi"] \\ M' \arrow[r, "\nabla'"'] & \Omega^1(A/k) \otimes_A M' \end{tikzcd}$$

Let $(A, \mathbf{g})$ be a Lie-Cartan pair over k. We define the *category of modules with integrable* $\mathbf{g}$-*connection* $\mathbf{MC}(A; \mathbf{g})$ in the following way: An object in $\mathbf{MC}(A; \mathbf{g})$ is a couple (M, ∇), where M is an A-module and ∇ is an integrable $\mathbf{g}$-connection on M. Given a pair of objects $(M, \nabla),\ (M', \nabla')$ in the category $\mathbf{MC}(A; \mathbf{g})$, we define a morphism from (M, ∇) to (M', ∇') to be a homomorphism $\phi : M \to M'$ of A-modules, such that the following diagram commuates for all elements $g \in \mathbf{g}$:

$$\begin{tikzcd} M \arrow[r, "\nabla_g"] \arrow[d, "\phi"'] & M \arrow[d, "\phi"] \\ M' \arrow[r, "\nabla'_g"'] & M' \end{tikzcd}$$

Let (M, ∇) be an object in the category $\mathbf{MC}(A)$, so $\nabla : M \to \Omega^1(A/k) \otimes_A M$ is an integrable connection on M. Let us consider the Lie-Cartan pair $(A; \mathbf{g})$ with $\mathbf{g} = \mathrm{Der}_k(A)$ and $\rho : \mathbf{g} \to \mathrm{Der}_k(A)$ the identity map. For each element $g \in \mathbf{g}$, $\rho_g \in \mathrm{Der}_k(A)$ corresponds to an A-linear homomorphism $\phi_g : \Omega^1(A/k) \to A$ by the universal property of the Kähler differentials. Let us denote by $\nabla'_g : M \to M$ the k-linear map given by $(\phi_g \otimes \mathrm{id}) \circ \nabla$. This clearly defines an A-linear map $\nabla' : \mathbf{g} \to \mathrm{End}_k(M)$, and since the connection ∇ has the derivation property, ∇' is a $\mathbf{g}$-connection on M.

Lemma 4.1. *There is a natural covariant functor $F : \mathbf{MC}(A) \to \mathbf{MC}(A; \mathrm{Der}_k(A))$.*

Proof. The assignment $(M, \nabla) \mapsto (M, \nabla')$, where ∇ is a connection on M and ∇' the corresponding covariant derivative on M, is functorial: This can easily be verified by considering the diagrams above. So the construction above induces a covariant functor. We have to prove that if the connection ∇ is integrable, then the covariant derivative ∇' is integrable as well: Let $g, h \in \mathrm{Der}_k(A)$ be derivations, and let us denote by $D(g \wedge h) : \Omega^2(A/k) \to A$ the A-linear map defined by the formula $D(g \wedge h)(da \wedge db) = g(a)h(b) - h(a)g(b)$ for all $a, b \in A$. A straight-forward calculation shows that $R_{\nabla'}(g \wedge h)$ is the composition $(D(g \wedge h) \otimes \mathrm{id}) \circ R_{\nabla}$. But $R_{\nabla} = 0$ since ∇ is integrable, so $R_{\nabla'} = 0$ and ∇' is integrable as well. $\square$

Let $(A, \mathbf{g})$ be a Lie-Cartan pair over k, and let V be the k-vector space given by $V = A \oplus \mathbf{g}$. Consider the tensor algebra $T(V)$ of V over k, defined as

$$T(V) = \bigoplus_{i \geq 0} V^i$$

with free multiplicative structure. This is an associative k-algebra, and we let J be the ideal generated by all relations of the form

i) $a * b - ab$,
ii) $a * g - ag$,
iii) $g * a - ag - \rho_g(a)$,
iv) $g * h - h * g - [g, h]$,

for all elements $a, b \in A$, $g, h \in \mathbf{g}$, where $*$ denotes the free multiplication in the tensor algebra $T(V)$. We denote by $R(A; \mathbf{g})$ the quotient $R(A; \mathbf{g}) = T(V)/J$, which is clearly an associative k-algebra.

Proposition 4.2. *Let $(A, \mathbf{g})$ be a Lie-Cartan pair over k. Then there is an equivalence of categories between $\mathbf{MC}(A; \mathbf{g})$ and the category $R(A; \mathbf{g})$-*Mod *of left $R(A; \mathbf{g})$-modules. In particular, the category $\mathbf{MC}(A; \mathbf{g})$ is Abelian.*

Proof. This follows directly from the construction of $R(A; \mathbf{g})$. $\square$

Consider the homomorphism $\rho : \mathbf{g} \to \mathrm{Der}_k(A)$ associated with the Lie-Cartan pair $(A, \mathbf{g})$. It induces a k-linear map $V \to \mathrm{D}^1(A)$, and consequently a k-algebra homomorphism $T(V) \to \mathrm{D}(A)$. We observe that the the ideal J maps to 0, and that the image of this map is contained in the derivation ring $\Delta(A) \subseteq \mathrm{D}(A)$. So we obtain a k-algebra homomorphism $\rho : R(A; \mathbf{g}) \to \Delta(A)$. We also see that if $\rho : \mathbf{g} \to \mathrm{Der}_k(A)$ is injective, then the ring homomorphism $\rho : R(A; \mathbf{g}) \to \Delta(A)$ is injective. In this case, we identify $R(A;\mathbf{g})$ with its image in $\Delta(A)$, and we write $\Delta(\mathbf{g})$ for this image.

Corollary 4.3. *Let $(A,\mathbf{g})$ be a Lie-Cartan pair over k such that $\rho : \mathbf{g} \to \mathrm{Der}_k(A)$ is injective. Then the category $\mathbf{MC}(A;\mathbf{g})$ is equivalent to the category of left $\Delta(\mathbf{g})$-modules. In particular, the category $\mathbf{MC}(A;\mathrm{Der}_k(A))$ is equivalent to the category of left $\Delta(A)$-modules, where $\Delta(A)$ is the derivation ring of A.*

3. Localizations

Let A be a commutative k-algebra, let S be a multiplicatively closed subset of A, and let M,N be A-modules. Then we have canonical homomorphism of $S^{-1}A$-modules

$$S^{-1}(M \otimes_A N) \to S^{-1}M \otimes_{S^{-1}A} S^{-1}N,$$
$$S^{-1}(\mathrm{Hom}_A(M,N)) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

The first is an isomorphism, and the second is an isomorphism if M is of finite presentation, see Bourbaki [**5**], proposition 2.2.7.18 and 2.2.7.19. Furthermore, we have a canonical homomorphism of $S^{-1}A$-modules $S^{-1}\Omega^1(A/k) \to \Omega^1(S^{-1}A/k)$, and this is an isomorphism by Matsumura [**24**], proposition 9.25.

Let $D \in \mathrm{Der}_k(A)$. Then there is a unique derivation $D' \in \mathrm{Der}_k(S^{-1}A)$ such that $D'(a/1) = D(a)/1$ for all $a \in A$: This derivation is given by the formula

$$D'(a/s) = (D(a)s - aD(s))/s^2$$

for all $a \in A,\ s \in S$. The uniqueness of D' follows since $D'(s/1 \cdot 1/s) = 0$ for all elements $s \in S$. We observe that if $d : \Omega^1(A/k) \to A$ is the universal derivation of A/k, then $d' : \Omega^1(S^{-1}A/k) \to S^{-1}A$ is the universal derivation of $S^{-1}A/k$. Clearly, the construction above defines a canonical homomorphism of $S^{-1}A$-modules $S^{-1}\mathrm{Der}_k(A) \to \mathrm{Der}_k(S^{-1}A)$. From the results in the previous paragraph, we see that this is an isomorphism if $\Omega^1(A/k)$ is an A-module of finite presentation. In particular, it is an isomorphism if A is a k-algebra of finite type.

Let $\nabla : M \to \Omega^1(A/k) \otimes_A M$ be a connection on M. We define a k-linear map $S^{-1}\nabla : S^{-1}M \to S^{-1}(\Omega^1(A/k) \otimes_A M)$ by the formula

$$S^{-1}\nabla(m/s) = 1/s^2(s\nabla(m) - d(s) \otimes m)$$

for all elements $s \in S,\ m \in M$. Let us show that this is a well-defined map. So assume that $um = 0$ for some $u \in S$. Then $u^2 \in S$, and we have

$$u^2(t\nabla(m) - d(t) \otimes m) = ut\nabla(um) - td(u) \otimes um - ud(t) \otimes um = 0.$$

Since $S^{-1}(\Omega^1(A/k) \otimes_A M) \cong \Omega^1(S^{-1}A/k) \otimes_{S^{-1}A} S^{-1}M$, we obtain a commutative diagram of k-vector spaces:

$$\begin{tikzcd}
M \arrow[r, "\nabla"] \arrow[d, "S^{-1}"'] & \Omega^1(A/k) \otimes_A M \arrow[d, "S^{-1}"] \\
S^{-1}M \arrow[r, "S^{-1}\nabla"'] & \Omega^1(S^{-1}A/k) \otimes_{S^{-1}A} S^{-1}M
\end{tikzcd}$$

It is straight-forward to see that $S^{-1}\nabla$ has the derivation property, since ∇ is a connection. In fact, $S^{-1}\nabla$ is the uniquely defined connection on $S^{-1}M$ such that the above diagram commutes.

Proposition 4.4. *Let A be a commutative k-algebra, and let $S \subseteq A$ be any multiplicatively closed subset. Then there exists a natural, covariant localization functor $S^{-1} : \mathbf{MC}(A) \to \mathbf{MC}(S^{-1}A)$.*

Proof. Clearly, the construction of $(S^{-1}M, S^{-1}\nabla)$ from (M, ∇) given above is functorial. So it is enough to show that $S^{-1}\nabla$ is integrable if ∇ is integrable. But a calculation shows that

$$R_{S^{-1}\nabla}(m/s) = 1/s\ R_\nabla(m)$$

for all elements $s \in S,\ m \in M$. So if $R_\nabla = 0$, then $R_{S^{-1}\nabla} = 0$ as well. $\square$

Let $S^{-1} : \mathrm{Der}_k(A) \to \mathrm{Der}_k(S^{-1}A)$ denote the composition of the A-linear localization map $\mathrm{Der}_k(A) \to S^{-1}\mathrm{Der}_k(A)$ with the canonical $S^{-1}A$-linear homomorphism $S^{-1}\mathrm{Der}_k(A) \to \mathrm{Der}_k(S^{-1}A)$ defined above. For any Lie-Cartan pair $(A, \mathbf{g})$ over k, let us denote by $S^{-1}\rho : S^{-1}\mathbf{g} \to \mathrm{Der}_k(S^{-1}A)$ the composition of the $S^{-1}A$-linear map $S^{-1}\rho : S^{-1}\mathbf{g} \to S^{-1}\mathrm{Der}_k(A)$ with the canonical $S^{-1}A$-linear homomorphism $S^{-1}\mathrm{Der}_k(A) \to \mathrm{Der}_k(S^{-1}A)$. We obtain the following commutative diagram of A-modules:

$$\begin{tikzcd}
\mathbf{g} \arrow[r, "\rho"] \arrow[d, "S^{-1}"'] & \mathrm{Der}_k(A) \arrow[d, "S^{-1}"] \\
S^{-1}\mathbf{g} \arrow[r] \arrow[dr, "S^{-1}\rho"'] & S^{-1}\mathrm{Der}_k(A) \arrow[d] \\
& \mathrm{Der}_k(S^{-1}A)
\end{tikzcd}$$

The $S^{-1}A$-module $S^{-1}\mathbf{g}$ has a natural Lie algebra structure over k, defined by the equation

$$[g/s, h/t] = [g,h]/st - \rho_g(t)h/st^2 + \rho_h(s)g/s^2t$$

for all $g, h \in \mathbf{g},\ s, t \in S$. A straight-forward calculation shows that the $S^{-1}A$-linear map $S^{-1}\rho : S^{-1}\mathbf{g} \to \mathrm{Der}_k(S^{-1}A)$ is a Lie algebra homomorphism over k.

Lemma 4.5. *Let $(A, \mathbf{g})$ be a Lie-Cartan pair over k, and let $S \subseteq A$ be a multiplicatively closed subset. Then $(S^{-1}A, S^{-1}\mathbf{g})$ is a Lie-Cartan pair over k as well. Assume that $\Omega^1(A/k)$ is an A-module of finite presentation. Then the structural homomorphism $S^{-1}\rho : S^{-1}\mathbf{g} \to \mathrm{Der}_k(S^{-1}A)$ is injective if $\rho : \mathbf{g} \to \mathrm{Der}_k(A)$ is injective.*

Proof. The first part is clear from the construction above. For the last part, it is clear that if ρ is injective, then $S^{-1}\mathbf{g} \to S^{-1}\mathrm{Der}_k(A)$ is injective as well. But if $\Omega^1(A/k)$ is of finite presentation, then $S^{-1}\mathrm{Der}_k(A) \to \mathrm{Der}_k(S^{-1}A)$ is an isomorphism, so the result follows. $\square$

Let M be an A-module, and let $\nabla : \mathbf{g} \to \mathrm{End}_k(M)$ be a $\mathbf{g}$-connection on M. Then, we define the $S^{-1}A$-linear map $S^{-1}\nabla : S^{-1}\mathbf{g} \to \mathrm{End}_k(S^{-1}M)$ by the formula

$$S^{-1}\nabla_{g/s}(m/t) = 1/st^2\ (t\nabla_g(m) - \rho_g(t)m)$$

for all $g \in \mathbf{g},\ s, t \in S,\ m \in M$. Clearly, $S^{-1}\nabla$ is a $S^{-1}\mathbf{g}$-connection on $S^{-1}M$, and the following diagram commutes for all $g \in \mathbf{g}$:

$$\begin{tikzcd}
M \arrow[r, "\nabla_g"] \arrow[d, "S^{-1}"'] & M \arrow[d, "S^{-1}"] \\
S^{-1}M \arrow[r, "S^{-1}\nabla_{g/1}"'] & S^{-1}M
\end{tikzcd}$$

Furthermore, $S^{-1}\nabla$ is the unique $S^{-1}\mathbf{g}$-connection on $S^{-1}M$ such that the above diagram commutes for all $g \in \mathbf{g}$.

Proposition 4.6. *Let $(A, \mathbf{g})$ be a Lie-Cartan pair over k, and let $S \subseteq A$ be a multiplicatively closed subset. Then there exists a natural, covariant localization functor $S^{-1} : \mathbf{MC}(A; \mathbf{g}) \to \mathbf{MC}(S^{-1}A; S^{-1}\mathbf{g})$.*

Proof. The construction of $(S^{-1}M, S^{-1}\nabla)$ from (M, ∇) is clearly functorial, so it is enough to show that $S^{-1}\nabla$ is integrable if ∇ is integrable. But a direct computation shows that

$$R_{S^{-1}\nabla}(g/s \wedge h/t)(m/u) = 1/stu \; R_{\nabla}(g \wedge h)(m)$$

for all $g, h \in \mathbf{g}, \; s, t, u \in S, \; m \in M$. So if $R_{\nabla} = 0$, then $R_{S^{-1}\nabla} = 0$ as well, which is what we wanted to prove. □

Let A be any commutative Noetherian k-algebra. Then A is regular if A_m is a regular local ring for all maximal ideals $m \subseteq A$. We know that $\Omega^1(A/k)$ is a locally free A-module if A is a regular k-algebra. This implies the following result:

Proposition 4.7. *Let A be a regular, commutative k-algebra of finite type. Then, the functor $F : \mathbf{MC}(A) \to \mathbf{MC}(A; \mathrm{Der}_k(A))$ is an equivalence of categories.*

Proof. We construct an inverse to the functor F. Let $\nabla' : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ be a covariant derivative on M. For all $m \in M$, we define $\phi_m \in \mathrm{Hom}_A(\mathrm{Der}_k(A), M)$ to be given by $\phi_m(D) = \nabla'_D(m)$. We clearly have $\mathrm{Der}_k(A) \cong \mathrm{Hom}_A(\Omega^1(A/k), A)$, so there is a natural A-linear map

$$\Omega^1(A/k) \otimes_A M \to \mathrm{Hom}_A(\mathrm{Der}_k(A), M),$$

given by $w \otimes m \mapsto \{D \mapsto D(w)m\}$ for all $w \in \Omega^1(A/k), \; m \in M$. But since A is a regular, commutative k-algebra of finite type, the localization of this map has the form $S^{-1}\Omega^1(A/k) \otimes_{S^{-1}A} S^{-1}M \to \mathrm{Hom}_{S^{-1}A}(\mathrm{Der}_k(S^{-1}A), S^{-1}M)$ for any multiplicatively closed subset S. Furthermore, we see that if $S = A \setminus p$ for some prime ideal $p \subseteq A$, this map is an isomorphism: This is clear, since $\Omega^1(A/k)$ and $\mathrm{Der}_k(A)$ are locally free A-modules with the same rank. So $\Omega^1(A/k) \otimes_A M$ and $\mathrm{Hom}_A(\mathrm{Der}_k(A), M)$ are naturally isomorphic A-modules. We define the k-linear map $\nabla : M \to \Omega^1(A/k) \otimes_A M$ by the formula $\nabla(m) = \phi_m$ for all $m \in M$. Then clearly, ∇ is a connection on M which the functor F maps to ∇'. Furthermore, we see that if ∇' is integrable, then ∇ is an integrable connection. □

4. Obstruction theory

Let A be a commutative k-algebra, and let M be an A-module. In this section, we shall describe the obstruction theory for connections on M, covariant derivatives on M, and more generally $\mathbf{g}$-connections on M for Lie-Cartan pairs $(A, \mathbf{g})$ over k with $\mathbf{g} \subseteq \mathrm{Der}_k(A)$. We shall describe this obstruction theory via Hochschild cohomology, and we refer to appendix A for definitions and elementary results on Hochschild cohomology.

We denote by $\psi \in \mathrm{Der}_k(A, \mathrm{Hom}_k(M, \Omega^1(A/k) \otimes_A M))$ the derivation given by $\psi(a)(m) = -da \otimes m$ for all $a \in A, \; m \in M$. From the definition of the Hochschild complex, this is a cocycle in $\mathrm{HC}^1(A, \mathrm{Hom}_k(M, \Omega^1(A/k) \otimes_A M))$, and it defines a canonical class $c(M) \in \mathrm{Ext}^1_A(M, \Omega^1(A/k) \otimes_A M)$ via Hochschild cohomology. The class $c(M)$ is called the *Atiyah-Kodaira-Spencer class* of M.

Proposition 4.8. *Let A be a commutative k-algebra and let M be an A-module. Then there is a canonical obstruction $c(M) \in \mathrm{Ext}^1_A(M, \Omega^1(A/k) \otimes_A M)$ such that $c(M) = 0$ if and only if there exists a connection on M. Furthermore, the set of all connections on M is a torsor over $\mathrm{Hom}_A(M, \Omega^1(A/k) \otimes_A M)$ if $c(M) = 0$.*

Proof. We have constructed $c(M)$ above. First, observe that $c(M) = 0$ if and only if there exists a connection on M: From the construction of $c(M)$, we have that $c(M) = 0$ if and only if $\psi = d^0(\nabla)$ for some $\nabla \in \mathrm{Hom}_k(M, \Omega^1(A/k) \otimes_A M)$. But ∇ is a connection on M if and only if $\nabla(am) = a\nabla(m) + da \otimes m$ for all $a \in A,\ m \in M$, or equivalently if $d^0(\nabla) = \psi$. Secondly, assume that $c(M) = 0$. Then we may choose a connection $\nabla : M \to \Omega^1(A/k) \otimes_A M$ on M. For any $\nabla' \in \mathrm{Hom}_k(M, \Omega^1(A/k) \otimes_A M)$, ∇' is a connection on M if and only if $\nabla' - \nabla \in \mathrm{Hom}_A(M, \Omega^1(A/k) \otimes_A M)$, so this proves the second part of the proposition. $\square$

For any $D \in \mathrm{Der}_k(A)$, let $\psi_D \in \mathrm{Der}_k(A, \mathrm{End}_k(M))$ be the derivation given by $\psi_D(a)(m) = -D(a)m$ for all $a \in A,\ m \in M$. We see that the derivation ψ_D maps to a uniquely defined element in $\mathrm{Ext}^1_A(M, M)$ via Hochschild cohomology, so we obtain a natural homomorphism

$$g : \mathrm{Der}_k(A) \to \mathrm{Ext}^1_A(M, M)$$

of k-vector spaces. We call this map the *Kodaira-Spencer map* of M, and we denote by $\mathbf{V} = \mathbf{V}(M) \subseteq \mathrm{Der}_k(A)$ its kernel, the *Kodaira-Spencer kernel* of M.

Lemma 4.9. *Let A be a commutative k-algebra, and let M be an A-module. Then $(A, \mathbf{V})$ is a Lie-Cartan pair over k.*

Proof. From the definition, it is clear that g is an A-linear homomorphism, so $\mathbf{V} \subseteq \mathrm{Der}_k(A)$ is clearly an A-submodule. It is therefore enough to show that it is a k-Lie subalgebra: For any derivation $D \in \mathrm{Der}_k(A)$, we see that $D \in \mathbf{V}$ if and only if $\psi_D = d^0(\phi)$ for some $\phi \in \mathrm{End}_k(M)$, where $d^0 : \mathrm{End}_k(M) \to \mathrm{Der}_k(A, \mathrm{End}_k(M))$ denotes the differential in the Hochschild complex. So if $D, D' \in \mathbf{V}$, then there exist $\phi, \phi' \in \mathrm{End}_k(M)$ with $d^0(\phi) = \psi_D,\ d^0(\phi') = \psi_{D'}$. An easy calculation shows that $d^0([\phi, \phi']) = \psi_{[D,D']}$ with $[\phi, \phi'] \in \mathrm{End}_k(M)$, so $[D, D'] \in \mathbf{V}$. $\square$

Proposition 4.10. *Let A be a commutative k-algebra, and let M be an A-module. Then there is a k-linear homomorphism $\nabla : \mathbf{V} \to \mathrm{End}_k(M)$ with the derivation property $\nabla_D(am) = a\nabla_D(m) + D(a)m$ for all $a \in A,\ m \in M,\ D \in \mathrm{Der}_k(A)$. Furthermore, $\mathbf{V}$ is the unique maximal k-linear subspace of $\mathrm{Der}_k(A)$ with this property.*

Proof. Let $D \in \mathrm{Der}_k(A)$. Then there exists an endomorphism $\nabla_D \in \mathrm{End}_k(M)$ with $\nabla_D(am) = a\nabla_D(m) + D(a)m$ for all $a \in A,\ m \in M$ if and only if $\psi_D = d^0(\nabla)$ for some $\nabla \in \mathrm{End}_k(M)$. But the last condition is equivalent with the condition that $D \in \mathbf{V}$, and the result follows. $\square$

Let $\nabla : \mathbf{V} \to \mathrm{End}_k(M)$ be a k-linear map with derivation property in the above sense. We define $l(\nabla)$ to be the k-linear map $l(\nabla) \in \mathrm{Hom}_k(A, \mathrm{Hom}_k(\mathbf{V}, \mathrm{End}_k(M)))$ given by

$$l(\nabla)(a)(D) = \nabla_{aD} - a\nabla_D$$

for all $a \in A,\ D \in \mathbf{V}$. An easy calculation shows that $\nabla_{aD} - a\nabla_D \in \mathrm{End}_A(M)$ for all $a \in A,\ D \in \mathbf{V}$, and that $l(\nabla) \in \mathrm{Hom}_k(A, \mathrm{Hom}_k(\mathbf{V}, \mathrm{End}_A(M)))$ is a derivation. So $l(\nabla)$ maps to a class $lc(M) \in \mathrm{Ext}^1_A(\mathbf{V}, \mathrm{End}_A(M))$, and this class is canonical in the sense that it does not depend upon the choice of k-linear map ∇.

Proposition 4.11. *Let A be a commutative k-algebra, and let M be an A-module. Then there exists a canonical obstruction $lc(M) \in \mathrm{Ext}^1_A(\mathbf{V}, \mathrm{End}_A(M))$ such that $lc(M) = 0$ if and only if there exists a $\mathbf{V}$-connection on M. Furthermore, the set of $\mathbf{V}$-connections on M is a torsor over $\mathrm{Hom}_A(\mathbf{V}, \mathrm{End}_A(M))$ if $lc(M) = 0$.*

Proof. We have constructed the obstruction $lc(M)$ above. First, let us show that $lc(M) = 0$ if and only if there is a $\mathbf{V}$-connection on M: Let $\nabla : \mathbf{V} \to \mathrm{End}_k(M)$ be a k-linear map with derivation property. If ∇ is A-linear, then $l(\nabla) = 0$, so clearly $lc(M) = 0$. Let us assume that $lc(M) = 0$. Then there exists a k-linear map $\phi : \mathbf{V} \to \mathrm{End}_A(M)$ such that $d^0(\phi) = l(\nabla)$. Let $\nabla' = \nabla + \phi$, then $\nabla' : \mathbf{V} \to \mathrm{End}_k(M)$ is A-linear and it has the derivation property by construction, so ∇' is a $\mathbf{V}$-connection on M. Secondly, let $lc(M) = 0$. Then we may choose a $\mathbf{V}$-connection $\nabla : \mathbf{V} \to \mathrm{End}_k(M)$ on M. For any A-linear map $\nabla' : \mathbf{V} \to \mathrm{End}_k(M)$, we see that ∇' is a $\mathbf{V}$-connection on M if and only if $\nabla' - \nabla \in \mathrm{Hom}_A(\mathbf{V}, \mathrm{End}_A(M))$, so the result follows. □

5. Connections on graded modules over monomial curves

Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let M be a $\mathbf{Z}$-graded A-module of finite type. Consider the multiplicatively closed subset $S \subseteq A$ consisting of all non-zero, homogeneous elements in A. Then we have seen that $S^{-1}A = T = k[t, t^{-1}]$. We observe that any homogeneous element in T is a unit. Clearly, $S^{-1}M = T \otimes_A M$ is a $\mathbf{Z}$-graded T-module of finite type. From the above comment, it follows that any minimal, homogeneous set of generators for $S^{-1}M$ is basis for $S^{-1}M$. So there is an isomorphism of $\mathbf{Z}$-graded T-modules $S^{-1}M \cong T^n$, where $n = \mathrm{rk}\, M = \dim_K(K \otimes_A M)$ and K is the field of fractions of A. In particular, $S^{-1}M = 0$ if and only if M is a torsion module.

Assume that M is a torsion free A-module. Then, the natural map of $\mathbf{Z}$-graded A-modules $M \to S^{-1}M$ is injective. Furthermore, $S^{-1}M \cong T^n$ for $n = \mathrm{rk}\, M > 0$. In this case, we shall identify M with its image in T^n, and consider M a $\mathbf{Z}$-graded A-submodule of T^n.

The case $n = 1$ is particularly simple, and we shall consider this case in more detail: Assume that M is a torsion free, $\mathbf{Z}$-graded A-module of rank 1. Then, we may consider M as a $\mathbf{Z}$-graded A-submodule of T by the above comment. Let $m_1, \dots, m_s$ be a minimal system of homogeneous generators for M considered as an A-module. Then we may choose $m_i = t^{d_i}$ with $d_i \in \mathbf{Z}$ for $1 \leq i \leq s$. Consider the set $\Lambda = \{\lambda \in \mathbf{Z} : t^\lambda \in M\}$, and let $k[\Lambda]$ be the k-linear subspace in T generated by $\{t^\lambda : \lambda \in \Lambda\}$. Then we have $M = k[\Lambda]$.

Proposition 4.12. *Let Γ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding monomial curve. For each set $\Lambda \subseteq \mathbf{Z}$ such that $\Gamma + \Lambda \subseteq \Lambda$, the k-linear subspace $k[\Lambda] \subseteq T$ is a torsion free, $\mathbf{Z}$-graded A-module of rank 1. In particular, the assignment $\Lambda \mapsto k[\Lambda]$ induces a bijective correspondence between the set $\{\Lambda : \Gamma \subseteq \Lambda \subseteq \mathbf{N_0},\ \Gamma + \Lambda \subseteq \Lambda\}$ and the set of equivalence classes of torsion free, $\mathbf{Z}$-graded A-modules of rank 1, up to graded isomorphisms of degree 0 and twists.*

Proof. The first part is clear from the construction above. For the second part, let us show that the correspondence $\Lambda \mapsto k[\Lambda]$ is surjective: Assume that M is a torsion free, $\mathbf{Z}$-graded A-module of rank 1. Then $M = k[\Lambda]$ for some set $\Lambda \subseteq \mathbf{Z}$ such that $\Gamma + \Lambda \subseteq \Lambda$. Let $d = \min \Lambda$. Since M is of finite type, this is a well-defined integer, so we may define $\Lambda' = \Lambda - d$. Clearly, $\Gamma + \Lambda' \subseteq \Lambda'$ and $\Gamma \subseteq \Lambda' \subseteq \mathbf{N_0}$. But we have $M[d] = k[\Lambda']$, and the surjectivity follows. The injectivity is clear: If $\Gamma \subseteq \Lambda, \Lambda' \subseteq \mathbf{N_0}$ and $k[\Lambda]$, $k[\Lambda']$ are equivalent, then there must be a homogeneous isomorphism of degree 0 between these graded A-modules. Hence $\Lambda = \Lambda'$. □

Let $(A, \mathbf{g})$ be a Lie-Cartan pair over k, and let M be an A-module. For any $\mathbf{g}$-connection $\nabla : \mathbf{g} \to \mathrm{End}_k(M)$, we know that $\nabla_g \in \mathrm{End}_k(M)$ is an endomorphism which satisfies $[\nabla_g, a] = \rho_g(a)$ for all $a \in A,\ g \in \mathbf{g}$. Since $\rho_g(a) \in \mathrm{End}_A(M)$ for all $a \in A$, we know that $\nabla_g \in \mathrm{D}^1_A(M)$ for all $g \in \mathbf{g}$.

Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let M be a torsion free, $\mathbf{Z}$-graded A-module of rank 1. Consider the multiplicatively closed subset $S \subseteq A$ consisting of the non-zero homogeneous elements in A. The localization $\mathrm{D}_A(M) \to S^{-1}\,\mathrm{D}_A(M)$ is injective, and we know that $S^{-1}\,\mathrm{D}_A(M) \cong \mathrm{D}_{S^{-1}A}(S^{-1}M) = \mathrm{D}(T)$ by Smith and Stafford [**32**], section 1.3 (d). Assume that $\nabla : \mathbf{g} \to \mathrm{End}_k(M)$ is a $\mathbf{g}$-connection on M, and let $g \in \mathbf{g}$. Then $\nabla_g \in \mathrm{D}_A(M)$ satisfies $[\nabla_g, a] = \rho_g(a)$ for all $a \in A$. We identify $\mathrm{D}_A(M)$ with its image in $\mathrm{D}(T)$. Then $\nabla_g = \rho_g + f_g$ for some $f_g \in T$ such that $(\rho_g + f_g) * M \subseteq M$. In particular, if $\mathbf{g} \subseteq \mathrm{Der}_k(A)$, then $\nabla_D = D + f_D$ for some $f_D \in T$ such that $(D + f_D) * M \subseteq M$ for all derivations $D \in \mathbf{g}$. Moreover, $f_g = \nabla_g(1)$ if $1 \in M$.

Corollary 4.13. *Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let M be a torsion free, $\mathbf{Z}$-graded A-module of rank 1. Then any covariant derivative on M is integrable.*

Proof. We may assume that $M = k[\Lambda]$, where $\Gamma \subseteq \Lambda \subseteq \mathbf{N_0}$ and $\Gamma + \Lambda \subseteq \Lambda$. Let $\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ be a covariant derivative, and let $f = \nabla_E(1) \in \mathrm{End}_A(M)$. We recall that $\mathrm{Der}_k(A)$ is generated as left A-module by $t^n E$ for $n \in \Gamma^{(1)}$. It is therefore enough to show that $R_\nabla(t^m E \wedge t^n E) = 0$ for $m, n \in \Gamma^{(1)} \cup \{0\}$. But $\nabla_{t^m E} = t^m \nabla_E$ in $\mathrm{D}(T)$ for all $m \in \mathbf{N_0}$: Let c be the conductor of Γ, then we see that $t^c \nabla_{t^m E} = t^c t^m \nabla_E$ in $\mathrm{D}(T)$, and $\nabla_{t^m E} = t^m \nabla_E$ since $\mathrm{D}(T)$ is an integral domain. A direct computation shows that $R_\nabla(t^m E \wedge t^n E) = 0$. □

We denote by $\mathrm{Hom}_A(\mathrm{Der}_k(A), \mathrm{End}_A(M))$ the *potentials* of M. Their importance derives from the fact that if ∇ is a covariant derivative on M, then the set of covariant derivatives on M is given by $\{\nabla + P : P \in \mathrm{Hom}_A(\mathrm{Der}_k(A), \mathrm{End}_A(M))$. Notice that if $A = k[\Gamma]$ is a monomial curve, and M is a torsion free, $\mathbf{Z}$-graded A-module of rank 1, we have a natural map $\mathrm{Hom}_A(\mathrm{Der}_k(A), \mathrm{End}_A(M)) \to \mathrm{End}_A(M)$ given by $P \mapsto P(E)$:

Proposition 4.14. *Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let M be a torsion free, $\mathbf{Z}$-graded A-module of rank 1. Then the map $\mathrm{Hom}_A(\mathrm{Der}_k(A), \mathrm{End}_A(M)) \to \mathrm{End}_A(M)$ given by $P \mapsto P(E)$ is injective, and its image is the set of all endomorphisms $f \in \mathrm{End}_A(M)$ such that $t^n f \in \mathrm{End}_A(M)$ for all $n \in \Gamma^{(1)}$.*

Proof. It is easy to see that the map is injective: Let P, P' be potentials of M. Since M is torsion free, and $P(E) = P'(E)$ implies $t^c P(t^n E) = t^c P'(t^n E)$ for all $n \in \Gamma^{(1)}$, where c is the conductor of Γ, the map is injective. It is also clear that if P is a potential, then $f = P(E)$ satisfies $t^n f \in \mathrm{End}_A(M)$ for all $n \in \Gamma^{(1)}$. Conversely, assume that $f \in \mathrm{End}_A(M)$ is such that $t^n f \in \mathrm{End}_A(M)$ for all $n \in \Gamma^{(1)}$. Then $P(t^n E) = t^n f$ defines a potential, so the result follows. □

Notice that the map $\mathrm{Hom}_A(\mathrm{Der}_k(A), \mathrm{End}_A(M)) \to \mathrm{End}_A(M)$ is a graded homomorphism of degree 0, so its image is a graded submodule of $\mathrm{End}_A(M)$. Let $f_0 \in \mathrm{End}_A(M)$ be a homogeneous element of degree 0. Then f_0 is in the image of the map above if and only if $f_0 = 0$ or $t^n \in \mathrm{End}_A(M)$ for all $n \in \Gamma^{(1)}$. We denote by $F \subseteq \mathrm{End}_A(M)$ the image of the above map, and by F_0 the homogeneous part of F of degree 0.

We define $\Lambda^{(0)} = \{w \in \mathbf{Z} : w + \Lambda \subseteq \Lambda\}$. Then it is easy to see that $\Lambda^{(0)} \subseteq \mathbf{N_0}$ is a numerical semigroup which satisfies $\Gamma \subseteq \Lambda^{(0)} \subseteq \Lambda$, and $\mathrm{End}_A(M)$ is the monomial curve $\mathrm{End}_A(M) = k[\Lambda^{(0)}]$. With this notation, we see that $F_0 = k$ if $\Gamma^{(1)} \subseteq \Lambda^{(0)}$, and $F_0 = 0$ otherwise. In particular, if Γ is a symmetric numerical semigroup, then $F_0 = k$ if $\Lambda \neq \Gamma$, and $F_0 = 0$ if $\Lambda = \Gamma$: To see this, we recall that $\Gamma^{(1)} = \{g\}$ when Γ is symmetric. If $\Lambda = \Gamma$, then clearly $g \notin \Lambda^{(0)}$ since $g + 0 = g \notin \Lambda$. If $\Lambda \neq \Gamma$, then there exists some $h \in \Lambda \setminus \Gamma$. Since Γ is symmetric and $h \notin \Gamma$, then $g - h \in \Gamma$, and consequently $g = (g - h) + h \in \Lambda$.

Proposition 4.15. *Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let $M = k[\Lambda]$ for $\Gamma \subseteq \Lambda \subseteq \mathbf{N_0}$ with $\Gamma + \Lambda \subseteq \Lambda$. Then there* *is an injective map from the set of covariant derivatives on* M *to* $\mathrm{End}_A(M)$*, given by* $\nabla \mapsto \nabla_E(1)$*. Furthermore, the image of this map is the set of endomorphisms* $f \in \mathrm{End}_A(M)$ *such that* $t^n(E+f) * M \subseteq M$ *for all* $n \in \Gamma^{(1)}$*.*

Proof. Let $\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ be a covariant derivative on M, and let $f = \nabla_E(1) \in M$. Then $\nabla_E = E + f$, considered as an element in $\mathrm{D}(T)$ by the comments above. Clearly, $E * t^n = nt^n$ for all $n \in \mathbf{Z}$, so $E * M \subseteq M$. It follows that $f * M \subseteq M$, so $f \in \mathrm{End}_A(M)$. We see that $\nabla \mapsto \nabla_E(1)$ induces a well-defined map with image in $\mathrm{End}_A(M) \subseteq T$. We will show that this map is injective: Assume that ∇, ∇' are covariant derivatives on M, with $\nabla_E(1) = \nabla'_E(1) = f$. We recall the following definition from section 2.5:

$$\Gamma^{(1)} = \{w \in \mathbf{Z} : \tau(w) = 1\} = \{w \in H : n + w \in \Gamma \text{ for all non-zero } n \in \Gamma\}$$

Since $\mathrm{Der}_k(A)$ is generated as left A-module by t^nE for $n \in \Gamma^{(1)}$, it is enough to prove that $\nabla_{t^nE} = \nabla'_{t^nE}$. We have $\nabla_E, \nabla'_E = E + f$ considered as elements in $\mathrm{D}(T)$, so $\nabla_E = \nabla'_E$. We obtain $t^c\nabla_{t^nE} = \nabla_{t^{n+c}E} = t^{n+c}\nabla_E = t^{n+c}\nabla'_E = t^c\nabla'_{t^nE}$ because $c, n+c \in \Gamma$, where c is the conductor of Γ. But M is torsion free, so this means that $\nabla_{t^nE} = \nabla'_{t^nE}$. For the last part, the condition $t^n(E+f) * M \subseteq M$ for all $n \in \Gamma^{(1)}$ is necessary, since $\nabla_{t^nE} = t^n(E+f)$. But assume that $f \in \mathrm{End}_A(M)$ is such that $t^n(E+f) * M \subseteq M$ for all $n \in \Gamma^{(1)}$. Then the formula $\nabla_{t^nE} = t^n(E+f)$ defines a covariant derivative $\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ with $\nabla_E(1) = f$, since $\mathrm{D}_A(M) = \{P \in \mathrm{D}(T) : P * M \subseteq M\}$. $\square$

Let $\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ be a covariant derivative on $M = k[\Lambda]$, and let $f \in \mathrm{End}_A(M)$ be given by $\nabla_E(1) = f$. Then f has a unique decomposition $f = f_0 + (f - f_0)$, where f_0 is homogeneous of degree 0, and $f_0, f - f_0 \in \mathrm{End}_A(M)$. Furthermore, $\nabla_E = (E + f_0) + (f - f_0)$. We have that $\nabla_{t^nE}(M) \subseteq M$ for all $n \in \Gamma^{(1)}$, so $t^n(E + f_0) * M \subseteq M$ and $t^n(f - f_0) * M \subseteq M$. This implies that $f - f_0 \in F$, and $P \in \mathrm{Hom}_A(\mathrm{Der}_k(A), \mathrm{End}_A(M))$ defined by $P(E) = f - f_0$ is a potential. We conclude that $\nabla' = \nabla - P$ is another covariant derivative, and $\nabla'_E = E + f_0$.

Theorem 4.16. *Let* Γ *be a numerical semigroup, and let* $A = k[\Gamma]$ *be the corresponding monomial curve. If* Γ *is symmetric, then there exists an integrable covariant derivative* $\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ *for all torsion free,* $\mathbf{Z}$*-graded* A*-modules* M *of rank* 1*.*

Proof. We may assume that $M = k[\Lambda]$ for $\Gamma \subseteq \Lambda \subseteq \mathbf{N_0}$ with $\Gamma + \Lambda \subseteq \Lambda$. Then clearly $\nabla_E = E$ defines a covariant derivative on M: It is enough to see that $E * M \subseteq M$ and that $t^gE * M \subseteq M$. We have seen that any covariant derivative is integrable, so this concludes the proof. $\square$

Notice that $\nabla_E = E + f_0$ defines an integrable, covariant derivative on M for all $f_0 \in k$ if $\Lambda \neq \Gamma$, while $\nabla_E = E$ defines the only integrable covariant derivative on M of this form if $\Lambda = \Gamma$.

Let $\Gamma =< 3, 4, 5 >$ and let $\Lambda = \Gamma \cup (\Gamma + 1)$. Then $M = k[\Lambda]$ is a torsion-free, $\mathbf{Z}$-graded module of rank 1 over the monomial curve $A = k[\Gamma]$. But there are no covariant derivatives on M: If there were, there would exist an $f_0 \in k$ such that $\nabla_E = E + f_0$ defined a covariant derivative. But $\Gamma^{(1)} = \{1, 2\}$, and $t(E + f_0) * t = (1 + f_0)t^2 \in M$, $t^2(E + f_0) * 1 = f_0t^2 \in M$, so $1 + f_0 = f_0 = 0$. This is impossible, so the condition that Γ is symmetric is necessary in the above theorem.

We remark that a torsion-free, $\mathbf{Z}$-graded module M of rank 1 over $A = k[\Gamma]$ is projective if and only if it is free. So M is projective if and only if $\Lambda = \Gamma$. Consequently, there exists integrable, covariant derivatives on many non-projective

A-modules M. In contrast, we mention the following result on the existence of D-module structures on M

Theorem 4.17. *Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let D be the ring of differential operators on A. For a torsion free, $\mathbf{Z}$-graded A-module M of rank 1, there exists a D-module structure on M compatible with the A-module structure if and only if M is projective.*

Proof. We may assume that $M = k[\Lambda]$, where $\Gamma \subseteq \Lambda \subseteq \mathbf{N_0}$ and $\Gamma + \Lambda \subseteq \Lambda$. Assume that $\rho : D \to \mathrm{D}_A(M)$ defines a D-module structure on M compatible with the A-module structure. Then $\rho(E) = E + f$ with $f \in \mathrm{End}_A(M)$. Let $m = t^\lambda \in M$, and let $P_w = t^w \chi_w(E) \in D$. Then $\rho(P_w)(m) = t^w \chi_w(E + f) * m \in M$, and the component of degree $w + \lambda$ is given by $t^w \chi_w(\lambda + f_0)m = \chi_w(\lambda + f_0)t^{\lambda+w}$. Since the component of degree 0 must be in M, we obtain $\lambda + w \in \Lambda$ or $\chi_w(\lambda + f_0) = 0$ for all $\lambda \in \Lambda$, $w \in \mathbf{Z}$. For $\lambda = 0$, this gives $\chi_w(f_0) = 0$ for all $w < 0$. We conclude that $f_0 = 0$. Assume that $\Lambda \neq \Gamma$. Then we may choose $\lambda \in \Lambda \setminus \Gamma$. If $\chi_w(\lambda + f_0) = \chi_w(\lambda) = 0$, then $\lambda \in \Omega(w)$. In particular, $\lambda \in \Gamma$ and this is impossible, so $w + \lambda \in \Lambda$ for all $w \in \mathbf{Z}$. But this is impossible, so $\Lambda = \Gamma$. It is clear that if $\Lambda = \Gamma$, then there exists a D-module structure on M compatible with the A-module structure, so this concludes the proof. $\square$

CHAPTER 5

Non-commutative deformation theory

In this chapter, we will introduce a non-commutative deformation theory for modules, which is due to Laudal (see Laudal [**21**], [**22**]). This theory generalizes the well-known local or formal deformation theory, as given by Schlessinger [**30**], in the case of deformations of modules: It will enable us to simultaneously deform a finite family of modules. We will state it for left modules (there is a similar theory for right modules), and we aim to make this introduction as self-contained as possible.

In the last section, we develop the theory of extensions of extensions of a finite family $\mathbf{M}$ of left R-module with fixed extension type G. In particular, we show that we can classify all such extensions of extensions via deformation theory. This theory is also due to Laudal, and is described in Laudal [**21**].

1. The categories $\mathbf{a}_p$ and $\hat{\mathbf{a}}_p$

Let p be a fixed natural number, and consider the ring k^p. This commutative ring has a natural k-algebra structure via the map $\alpha \mapsto (\alpha, \dots, \alpha)$ for $\alpha \in k$, and we shall always consider it as a k-algebra via this map. We denote the ideal $\mathrm{pr}_i(k^p) \subseteq k^p$ by k_i for $1 \leq i \leq p$, where $\mathrm{pr}_i : k^p \to k^p$ is the i'th projection, and consider k_i as a k^p-module. Clearly, k^p is an Artinian ring, and $\{k_1, \dots, k_p\}$ is the set of isomorphism classes of simple k^p-modules.

A *p-pointed k-algebra* is a triple (S, f, g), where S is an associative ring, and $f : k^p \to S,\ g : S \to k^p$ are ring homomorphisms such that $g \circ f = \mathrm{id}$. A morphism $u : (S, f, g) \to (S', f', g')$ of p-pointed k-algebras is a ring homomorphism $u : S \to S'$ such that the natural diagrams commute. That is, such that $u \circ f = f'$ and $g' \circ u = g$. We shall denote the category of p-pointed k-algebras by $\mathbf{A}_p$. Notice that if (S, f, g) is an object of $\mathbf{A}_p$, then f is injective and g is surjective, and we shall identify k^p with its image in S. We often write S for the object (S, f, g) to simplify notation.

Let (S, f, g) be an object in $\mathbf{A}_p$, and denote this object by S. We define the *radical* of S to be $I(S) = \ker(g)$, which is an ideal in S. Furthermore, we denote by $J(S)$ the *Jacobson radical* of S. This radical is defined by

$$J(S) = \{x \in S : xM = 0 \text{ for all simple left } S\text{-modules } M\}$$

and it is also an ideal in S (see Lam [**18**], Corollary 4.2). We shall write I, J for the radicals $I(S), J(S)$ when the meaning of these expressions is clear from the context. Notice that the Jacobson radical J depends only on the ring S, while the radical I depends on the structural morphism g as well.

For all objects S in $\mathbf{A}_p$, the inclusion $J(S) \subseteq I(S)$ holds: We have $J(k^p) = 0$ since k^p is semi-simple, and $g(J(S)) \subseteq J(k^p) = 0$ since g is a surjection (see Anderson, Fuller [**1**], Corollary 15.8). In general, we know that S and $S/J(S)$ have the same simple left modules (see Lam [**18**], proposition 4.8). So if we consider k_i as a left S-module via the morphism $g : S \to k^p$ for $1 \leq i \leq p$, we see that $\{k_1, \dots, k_p\}$ is contained in the set of isomorphism classes of simple left S-modules, and the equality $J(S) = I(S)$ holds if and only if $\{k_1, \dots, k_p\}$ is the full set of isomorphism classes of simple left S-modules. Consequently, it is clear that the equality $I(S) = J(S)$ does not hold in general: For a counter-example, consider

$S = k[x]/(x - x^2)$ with the natural k-algebra structure $f : k \to S$ and let $g : S \to k$ be given by $x \mapsto 0$. Then S is an object of $\mathbf{A}_1$, but $J(S) \neq I(S)$ because S has two non-isomorphic simple left S-modules (given by $x \mapsto 0$ and $x \mapsto 1$).

We denote by e_i the idempotent $(0, 0, \dots, 1, \dots, 0) \in k^p$ for $1 \leq i \leq p$. Notice that $e_i e_j = 0$ if $i \neq j$, and that $e_1 + \cdots + e_p = 1$. For any object S in $\mathbf{A}_p$, we identify $\{e_1, \dots, e_p\}$ with idempotents in S via the inclusion $k^p \to S$. Denote by S_{ij} the k-linear sub-space $e_i S e_j \subseteq S$. We immediately see, using the properties of the idempotents, that the following relations hold for $1 \leq i, j, l, m \leq p$:

1. $S_{ij} S_{lm} \subseteq \delta_{jl} S_{im}$,
2. $S_{ij} \cap S_{lm} = 0$ if $(i, j) \neq (l, m)$,
3. $\sum S_{ij} = S$.

In particular, we have that $S = \oplus S_{ij}$, so every element $s \in S$ may be written in matrix form $s = (s_{ij})$ with $s_{ij} \in S_{ij}$ for $1 \leq i, j \leq p$. Furthermore, elements of S also multiply as matrices. It is therefore reasonable to call an object S in $\mathbf{A}_p$ a matrix ring, and to write it $S = (S_{ij})$. Notice that S_{ii} is an associative ring (with identity e_i), and that S_{ij} is a (unitary) $S_{ii} - S_{jj}$ bimodule for $1 \leq i, j \leq p$. For any ideal $K \subseteq S$, we see that $e_i K e_j = K \cap S_{ij}$, and we shall denote this k-linear subspace K_{ij} for $1 \leq i, j \leq p$. Since $K = \oplus K_{ij}$, we write $K = (K_{ij})$.

We recall that an associative ring is *Artinian* (*Noetherian*) if and only if it is left and right Artinian (Noetherian). That is, if and only if the ring has the DCC (ACC) for left ideals and for right ideals. We would like to have methods to decide when an associative ring is Artinian or Noetherian. For objects in $\mathbf{A}_p$, we state the following useful proposition:

Proposition 5.1. *Let $S = (S_{ij})$ be an object in $\mathbf{A}_p$. Then S is Noetherian (Artinian) if and only if the following conditions hold:*
i) S_{ii} is Noetherian (Artinian) for $1 \leq i \leq p$,
ii) S_{ij} is a Noetherian (Artinian) left S_{ii}-module and a Noetherian (Artinian) right S_{jj}-module for $1 \leq i, j \leq p$, $i \neq j$.

Proof. Let $C_j = \oplus S_{ij}$ be the j'th column of S for $1 \leq j \leq p$. Then each C_j is a left S-module, and S is left Noetherian (Artinian) if and only if C_j is a Noetherian (Artinian) left S-module for $1 \leq j \leq p$. Furthermore, we see that C_j is a Noetherian (Artinian) left S-module if and only if S_{ij} is a Noetherian (Artinian) left S_{ii}-module for $1 \leq i \leq p$. It follows that S is left Noetherian (Artinian) if and only if S_{ij} is a Noetherian (Artinian) left S_{ii}-module for $1 \leq i, j \leq p$. On the other hand, we may consider the i'th row of S, $R_i = \oplus S_{ij}$ for $1 \leq i \leq p$. A similar argument shows that S is right Noetherian (Artinian) if and only if S_{ij} is a Noetherian (Artinian) right S_{jj}-module for $1 \leq i, j \leq p$. $\square$

We recall that a finitely generated, associative k-algebra is not necessarily Noetherian. That is, Hilbert's basis theorem does not hold for associative rings. For a counter-example, let $S = k\{x_1, \dots, x_n\}$ be the free, associative k-algebra on n generators. It is well-known that S is Noetherian only if $n = 1$. However, we know from the Hopkins-Levitzki theorem (see Lam [**18**], Theorem 4.15) that an associative Artinian ring is Noetherian.

A k-algebra S is Artinian if S has finite dimension as k-vector space. This is clear, since S is an Artinian k-vector space if and only if S has finite dimension as k-vector space. We have a converse statement under certain conditions:

Lemma 5.2. *Let S be an object of $\mathbf{A}_p$, and let $I = I(S)$. If S is Artinian and $I^n = 0$ for some $n \geq 1$, then S has finite dimension as a k-vector space.*

Proof. Since S is Artinian, it is Noetherian, and I^m is finitely generated as left S-module for all m. Consequently, I^m / I^{m+1} is a finitely generated left S/I-module for all m, and hence of finite dimension as a k-vector space. But $I^n = 0$, and in particular a k-vector space of finite dimension. By induction, we see that I^m has finite dimension as a k-vector space for all m, and consequently, S has finite dimension as a k-vector space. □

We define the *category* $\mathbf{a}_p$ to be the full sub-category of $\mathbf{A}_p$ consisting of objects S in $\mathbf{A}_p$ such that S is Artinian and $I(S) = J(S)$. The condition $I(S) = J(S)$ might equivalently be replaced by the condition that $I(S)$ is a nilpotent ideal (that is, that $I(S)^n = 0$ for some $n \geq 1$), since the Jacobson radical is the largest nilpotent ideal in an Artinian ring (see Lam [**18**], Theorem 4.12). In particular, all objects S in $\mathbf{a}_p$ have finite dimension as k-vector spaces by lemma 5.2. From the comment above, we also have a geometric interpretation of the condition $I(S) = J(S)$: It means that $\{k_1, \dots, k_p\}$ is the set of isomorphism classes of simple left S-modules, or equivalently, that the number of isomorphism classes of simple left S-modules is p.

Let S be an object in $\mathbf{A}_p$, with radical $I = I(S)$. Then the I-adic filtration defines a topology on S compatible with the ring operations (see Bourbaki [**5**], chapter III, §2, no. 5), and we shall always consider S a topological ring in this way. We say that the topology on S is Hausdorff (or separated) if and only if $\cap I^n = 0$.

For all objects S in $\mathbf{A}_p$, there is an I-adic completion $\hat{S}$ of S and a canonical morphism $S \to \hat{S}$ in $\mathbf{A}_p$. The I-adic completion $\hat{S}$ is defined by the projective limit

$$\hat{S} = \varprojlim S/I^n,$$

and the morphism $S \to \hat{S}$ is the natural one induced by this projective limit. We say that S is *complete* (or separated complete) if the natural morphism $S \to \hat{S}$ is an isomorphism in $\mathbf{A}_p$. Since $S/I^n \cong \hat{S}/I(\hat{S})^n$ for all $n \geq 0$, any I-adic completion is complete. We remark that S is complete if and only if S is topologically complete and has a Hausdorff topology.

We define the *pro-category* $\hat{\mathbf{a}}_p$ of $\mathbf{a}_p$ to be the full sub-category of $\mathbf{A}_p$ consisting of objects S in $\mathbf{A}_p$ such that S is complete and $S/I(S)^n$ belongs to $\mathbf{a}_p$ for all $n \geq 1$. In particular, all objects S in $\hat{\mathbf{a}}_p$ have a topology which is Hausdorff. We also see that there is an inclusion of categories $\mathbf{a}_p \subseteq \hat{\mathbf{a}}_p$.

Let S be an object in $\hat{\mathbf{a}}_p$, and let $I = I(S)$. To fix notation, we denote by $\operatorname{gr}_n S$ the k-vector space I^n/I^{n+1} for $n \geq 0$ (with $I^0 = S$), and denote by $\operatorname{gr} S = \oplus \operatorname{gr}_n S$ the graded ring associated to the I-adic filtration of S. Furthermore, we define the tangent space of S to be the k-linear space dual to $\operatorname{gr}_1 S$,

$$t_S = \operatorname{Hom}_k(I/I^2, k) = (I/I^2)^*,$$

which is clearly of finite dimension. In particular, we have $(t_S)^* \cong I/I^2$.

Let $u : S \to T$ be a morphism in $\hat{\mathbf{a}}_p$. Since u preserves the given filtrations of S and T, i.e. $u(I(S)^n) \subseteq I(T)^n$ for all $n \geq 0$, u induces a morphism of graded rings $\operatorname{gr}(u) : \operatorname{gr} S \to \operatorname{gr} T$. Since $\operatorname{gr}(u)$ is homogeneous of degree 0, u also induces morphisms of k-vector spaces $\operatorname{gr}_n(u) : \operatorname{gr}_n S \to \operatorname{gr}_n T$ for all $n \geq 0$. In particular, we have a morphism of k-vector spaces $\operatorname{gr}_1(u) : \operatorname{gr}_1 S \to \operatorname{gr}_1 T$, and a dual morphism $t_u : t_T \to t_S$.

Proposition 5.3. *Let $u : S \to T$ be a morphism in $\hat{\mathbf{a}}_p$. Then u is a surjection if and only if $\operatorname{gr}_1(u)$ is a surjection. Furthermore, u is injective if $\operatorname{gr}(u)$ is injective.*

Proof. If u is surjective, then clearly $\operatorname{gr}_1(u)$ is also surjective. To prove the other implication, let us consider the map $\operatorname{gr}(u) : \operatorname{gr}(S) \to \operatorname{gr}(T)$. Since $\operatorname{gr} T$ is generated by the elements in $\operatorname{gr}_1 T$ as an algebra, it follows that if $\operatorname{gr}_1(u)$ is surjective, then $\operatorname{gr}(u)$ is also surjective. From Bourbaki [**5**], chapter III, §2, no. 8, corollary 1 and

2, we have that u is surjective (injective) if $\mathrm{gr}(u)$ is surjective (injective), and the result follows. $\square$

Let n be any natural number. We define *the category* $\mathbf{a}_p(n)$ to be the full sub-category of $\mathbf{a}_p$ consisting of objects S in $\mathbf{a}_p$ such that $I(S)^n = 0$. Notice that $\mathbf{a}_p(n) \subseteq \mathbf{a}_p(n+1)$ for all $n \geq 1$. Furthermore, each object S in $\mathbf{a}_p$ belongs to a sub-category $\mathbf{a}_p(n)$ for some integer n.

We conclude this section with an important family of examples: Let V_{ij} be a finite dimensional k-vector space for $1 \leq i, j \leq p$, with $\dim_k V_{ij} = d_{ij}$. Let furthermore $\{s_{ij}(l) : 1 \leq l \leq d_{ij}\}$ be a basis of V_{ij} for $1 \leq i, j \leq p$ (or simply $\{s_{ij}\}$ if $d_{ij} = 1$). We define the *free matrix ring* $S = S(\{V_{ij}\})$ defined by the vector spaces V_{ij} in the following way: We say that a monomial in S of type (i, j) and degree n is an expression of the form

$$s_{m_0 m_1}(l_1) s_{m_1 m_2}(l_2) \dots s_{m_{n-1} m_n}(l_n)$$

with $m_0 = i, m_n = j$. To these, we add the monomials e_i for $1 \leq i \leq p$, which we consider to be of type (i, i) and degree 0. We define S to be the k-linear space generated by all monomials in S, with the obvious multiplication: Let M be a monomial of type (i, j), and M' a monomial of type (l, m). If $j \neq l$, then $MM' = 0$, and otherwise the product is the new monomial obtained by juxtapositioning M and M' (possibly after having erased unnecessary e_i's). We see that S is an object of the category $\mathbf{A}_p$, via the obvious maps $k^p \to S \to k^p$, and S_{ij} is the k-linear subspace generated by monomials in S of type (i, j). The ideal $I = I(S)$ is the k-linear subspace generated by all monomials of positive degree.

We denote by $\hat{S} = \hat{S}(\{V_{ij}\})$ the completion of S, and call this the *formal matrix ring* defined by the vector spaces V_{ij}. Explicitly, every element in $\hat{S}_{ij}$ is an infinite k-linear sum of monomials in S of type (i, j). Let $I = I(S)$. We have that $S_n = S/I^n \cong \hat{S}/I(\hat{S})^n$ belongs to $\mathbf{a}_p$ for $n \geq 1$: Clearly, S_n has finite dimension as k-vector space, so S_n is Artinian, and $I(S_n) = I/I^n$, so the radical is nilpotent. Since $\hat{S}$ clearly is complete, it follows that $\hat{S}$ belongs to $\hat{\mathbf{a}}_p$.

Notice that neither S nor $\hat{S}$ is Noetherian in general. For a counter-example, it is enough to consider the case when $p = 2$ and $d_{11} = d_{12} = d_{21} = 1,\ d_{22} = 0$. In this case, $S_{11} = k\{s_{11}, s_{12}s_{21}\} \cong k\{x, y\}$, which is not Noetherian. Similarly, $\hat{S}_{11} \cong k\{\{x, y\}\}$, which is not Noetherian. So by proposition 5.1, neither S nor $\hat{S}$ is Noetherian in this case.

2. Non-commutative deformation functors

Let R be a fixed associative k-algebra, and let $\mathbf{M} = \{M_1, \dots, M_p\}$ be a finite family of left R-modules. In this section, we shall define a deformation functor

$$\mathrm{Def}_{\mathbf{M}} : \mathbf{a}_p \to \mathbf{Sets}$$

describing how these R-modules can be deformed simultaneously.

Let S be an object of $\mathbf{a}_p$. A lifting of the family $\mathbf{M}$ of left R-modules to S is a left $R \otimes_k S^{\mathrm{op}}$-module M_S, together with isomorphisms of left R-modules $\eta_i : M_S \otimes_S k_i \to M_i$ for $1 \leq i \leq p$, such that $M_S \cong (M_i \otimes_k S_{ij})$ as right S-modules. We remark that a left $R \otimes_k S^{\mathrm{op}}$-module is the same as an R-S bimodule such that the left and right k-vector space structures coincide. Furthermore, the notation $(M_i \otimes_k S_{ij})$ refers to the k-vector space

$$(M_i \otimes_k S_{ij}) = \bigoplus_{i,j} (M_i \otimes_k S_{ij})$$

with the natural right S-module structure coming from the multiplication in S. The condition that $M_S \cong (M_i \otimes_k S_{ij})$ as right S-modules replaces the flatness condition in the theory of local deformations, and we shall refer to it as the flatness condition in this theory as well.

Let M'_S, M''_S be two liftings of $\mathbf{M}$ to S. We say that these two liftings are equivalent if there exists an isomorphism $\tau : M'_S \to M''_S$ of left $R \otimes_k S^{\mathrm{op}}$-modules such that the natural diagrams commute. That is, such that $\eta''_i \circ (\tau \otimes_S k_i) = \eta'_i$ for $1 \leq i \leq p$. We let $\mathrm{Def}_{\mathbf{M}}(S)$ denote the set of equivalence classes of liftings of $\mathbf{M}$ to S, and we refer to these equivalence classes as *deformations* of $\mathbf{M}$ to S. We shall often denote a deformation represented by (M_S, η_i) by M_S to simplify notation.

Let $u : S \to T$ be a morphism in $\mathbf{a}_p$, and let M_S be a lifting of $\mathbf{M}$ to S, representing an element in $\mathrm{Def}_{\mathbf{M}}(S)$. We define $M_T = M_S \otimes_S T$, which has a natural structure as a left $R \otimes_k T^{\mathrm{op}}$-module. Since u is a morphism in $\mathbf{a}_p$, we have natural isomorphisms of left R-modules

$$(M_S \otimes_S T) \otimes_T k_i \cong M_S \otimes_S k_i,$$

inducing isomorphisms of left R-modules $\rho_i : M_T \otimes_T k_i \to M_i$ via η_i for $1 \leq i \leq p$. A straight-forward calculation shows that M_T together with the isomorphisms ρ_i for $1 \leq i \leq p$ constitutes a lifting of $\mathbf{M}$ to T, and furthermore that the equivalence class of this lifting is independent upon the representative of the equivalence class of M_S. Hence, we obtain a map $\mathrm{Def}_{\mathbf{M}}(u) : \mathrm{Def}_{\mathbf{M}}(S) \to \mathrm{Def}_{\mathbf{M}}(T)$, and we see that $\mathrm{Def}_{\mathbf{M}} : \mathbf{a}_p \to \mathbf{Sets}$ is a covariant functor.

Let $S = (S_{ij})$ be an object in $\mathbf{a}_p$. We shall give a more explicit way of calculating $\mathrm{Def}_{\mathbf{M}}(S)$: We may assume that every element of $\mathrm{Def}_{\mathbf{M}}(S)$ is represented by a lifting M_S, such that $M_S = (M_i \otimes_k S_{ij})$ considered as a right S-module. In order to describe this lifting completely, it is enough to describe the left action of R on M_S. Furthermore, it is enough to describe this action on elements of the form $m_i \otimes e_i$ with $m_i \in M_i$, since we have

$$r(m_i \otimes s_{ij}) = (r(m_i \otimes e_i))s_{ij}$$

for all $r \in R,\ m_i \in M_i,\ s_{ij} \in S_{ij}$. For a fixed $r \in R,\ m_i \in M_i$, assume that $r(m_i \otimes e_i) = \sum(m'_j \otimes s'_{jl})$ with $m'_j \in M_j,\ s'_{jl} \in S_{jl}$. Then multiplication by e_i on the right gives the equality

$$r(m_i \otimes e_i) = \sum_j (m'_j \otimes s'_{ji}),$$

and the isomorphism η_i gives a further restriction on the left action of R, expressed by the formula

$$r(m_i \otimes e_i) = (rm_i) \otimes e_i + \sum_j m'_j \otimes s'_{ji}, \tag{3}$$

where $r \in R,\ m_i \in M_i,\ m'_j \in M_j,\ s'_{ji} \in I(S)_{ji}$. Consequently, the set $\mathrm{Def}_{\mathbf{M}}(S)$ consists of all possible ways of choosing a left R-action on elements of the form $m_i \otimes e_i$, fulfilling the above conditions and the associativity condition, up to equivalence.

Let S be any object in $\mathbf{a}_p$. Then the formula $r(m_i \otimes e_i) = (rm_i) \otimes e_i$ for $r \in R,\ m_i \in M_i$ defines a left R-module structure on $(M_i \otimes S_{ij})$ compatible with the right S-module structure. Hence, there exists a trivial lifting M_S to S for all objects S in $\mathbf{a}_p$, and $\mathrm{Def}_{\mathbf{M}}(S)$ is non-empty. Notice that in the case $S = k^p$, we have $I = I(S) = 0$, such that this trivial lifting is the only one possible. Consequently, we have $\mathrm{Def}_{\mathbf{M}}(k^p) = \{*\}$, where $*$ denotes the equivalence class of the trivial lifting.

Let $u : S \to T$ be a morphism in $\mathbf{A}_p$, and let $M_T \in \mathrm{Def}_{\mathbf{M}}(T)$ be a given deformation. We say that a deformation $M_S \in \mathrm{Def}_{\mathbf{M}}(S)$ is a lifting of M_T or is lying over M_T if $\mathrm{Def}_{\mathbf{M}}(u)(M_S) = M_T$. Given any object S in $\mathbf{A}_p$ and a deformation $M_S \in \mathrm{Def}_{\mathbf{M}}(S)$, we see that M_S is a lifting of the trivial deformation $*$ in $\mathrm{Def}_{\mathbf{M}}(k^p)$ in the above sense, relative to the structural morphism $g : S \to k^p$. Hence, our notation is consistent.

For another example, consider the test algebras $S(m,n)$ for $1 \leq m,n \leq p$, constructed in the following way: Let S be the free matrix algebra defined by the k-vector spaces V_{ij} with $d_{ij} = 1$ if $(i,j) = (m,n)$, and $d_{ij} = 0$ otherwise. Then we define $S(m,n) = S/I(S)^2$, and this is an object in $\mathbf{a}_p$ by construction. Let $\xi \in \mathrm{Ext}^1_R(M_n, M_m)$, then there exists a derivation $\psi \in \mathrm{Der}_k(R, \mathrm{Hom}_k(M_n, M_m))$ representing ξ in Hochschild cohomology (see appendix A for a general reference of Hochshild cohomology). We denote by $M(\psi)$ the lifting of $\mathbf{M}$ to $S(m,n)$ given by the left R-action

$$r(m_i \otimes e_i) = (rm_i) \otimes e_i + \delta_{in}\psi(r)(m_n) \otimes \varepsilon_{mn}$$

with $r \in R,\ m_i \in M_i,\ 1 \leq i \leq p$, and where ε_{mn} is the equivalence class of s_{mn} in $S(m,n)$. A straight-forward calculation shows that the liftings $M(\psi),\ M(\psi')$ are equivalent if and only if $\psi, \psi' \in \mathrm{Der}_k(R, \mathrm{Hom}_k(M_n, M_m))$ define the same element in $\mathrm{Ext}^1_R(M_n, M_m)$. Furthermore, we see that any lifting of $\mathbf{M}$ to $S(m,n)$ is of the form $M(\psi)$ with $\psi \in \mathrm{Der}_k(R, \mathrm{Hom}_k(M_n, M_m))$, up to equivalence. Hence, the map $\psi \mapsto M(\psi)$ induces an isomorphism of sets between $\mathrm{Ext}^1_R(M_n, M_m)$ and $\mathrm{Def}_{\mathbf{M}}(S(m,n))$. In particular, $\mathrm{Def}_{\mathbf{M}}(S(m,n))$ has a natural k-vector space structure inherited from $\mathrm{Ext}^1_R(M_n, M_m)$.

3. Pro-representing hulls of functors on $\mathbf{a}_p$

Let $\mathrm{F} : \mathbf{a}_p \to \mathbf{Sets}$ be a functor. In this section, we shall consider such functors which are pointed, that is, such that $\mathrm{F}(k^p) = \{*\}$. Notice that any functor F on $\mathbf{a}_p$ may be extended to a functor $\hat{\mathrm{F}} : \hat{\mathbf{a}}_p \to \mathbf{Sets}$ on the pro-category $\hat{\mathbf{a}}_p$ by the formula

$$\hat{\mathrm{F}}(S) = \varprojlim \mathrm{F}(S/I^n)$$

for objects S in $\hat{\mathbf{a}}_p$ with $I = I(S)$. Furthermore, any functor F on $\mathbf{a}_p$ has a restriction $\mathrm{F}_n : \mathbf{a}_p(n) \to \mathbf{Sets}$ to the sub-category $\mathbf{a}_p(n)$.

Let R be an associative k-algebra, and $\mathbf{M}$ a finite family of left R-modules. As we have seen in the previous section, $\mathrm{Def}_{\mathbf{M}}$ is an example of a pointed functor on $\mathbf{a}_p$. For another example, let S be an object in $\hat{\mathbf{a}}_p$. We define h_S to be the functor $\mathrm{Mor}(S,-) : \mathbf{a}_p \to \mathbf{Sets}$, where Mor denotes the morphisms in the pro-category $\hat{\mathbf{a}}_p$. Since $\mathrm{h}_S(k^p) = \{*\}$, this is a also a pointed functor on $\mathbf{a}_p$.

Lemma 5.4. *Let S be an object in $\hat{\mathbf{a}}_p$, and let $\mathrm{F} : \mathbf{a}_p \to \mathbf{Sets}$ be a functor. Then there is a natural isomorphism of sets*

$$\alpha : \hat{\mathrm{F}}(S) \to \mathrm{Mor}(\mathrm{h}_S, \mathrm{F}),$$

where Mor *denotes the morphisms in the category of functors from $\mathbf{a}_p$ to* **Sets**.

Proof. Let $\xi \in \hat{\mathrm{F}}(S)$, then $\xi = (\xi_n)$ with $\xi_n \in \mathrm{F}(S/I^n)$ for all $n \geq 1$. For any object T in $\mathbf{a}_p$, we construct a map of sets $\alpha(\xi)_T : \mathrm{Mor}(S,T) \to \mathrm{F}(T)$: Let $u : S \to T$ be a morphism in $\hat{\mathbf{a}}_p$, then $u(I(S)) \subseteq I(T)$, and $I(T)$ is nilpotent since T is in $\mathbf{a}_r$, so there exists $n \geq 1$ such that u factorizes through $u_n : S/I(S)^n \to T$. We define $\alpha(\xi)_T(u) = \mathrm{F}(u_n)(\xi_n)$, and a straight-forward calculation shows that this expression is independent upon the choice of n, and gives rise to a natural transformation of functors. Conversely, let $\phi : \mathrm{h}_S \to F$ be a natural transformation of functors on $\mathbf{a}_p$. Then we define $\xi_n \in \mathrm{F}(S/I(S)^n)$ to be $\xi_n = \phi_{S/I(S)^n}(S \to S/I(S)^n)$, where $S \to S/I(S)^n$ is the natural morphism. Again, a straight-forward calculation shows that $\xi = (\xi_n)$ defines an element in $\hat{\mathrm{F}}(S)$, and that this map of sets defines an inverse to α. □

There is also a version of lemma 5.4 for the category $\mathbf{a}_p(n)$: For an object S in $\mathbf{a}_p(n)$, and a functor $\mathrm{F} : \mathbf{a}_p(n) \to \mathbf{Sets}$, there is a natural isomorphism of sets

$$\alpha_n : \mathrm{F}(S) \to \mathrm{Mor}(\mathrm{h}_S, \mathrm{F}),$$

where Mor denotes the morphisms in the category of functors from $\mathbf{a}_p(n)$ to **Sets**. The construction of this isomorphism is similar to the construction in lemma 5.4.

We recall that a morphism $\phi : \mathrm{F} \to \mathrm{G}$ of functors $\mathrm{F}, \mathrm{G} : \mathbf{a}_p \to \mathbf{Sets}$ is *smooth* if the following condition holds: For all surjective morphisms $u : S \to T$ in $\mathbf{a}_p$, the natural map of sets

$$\mathrm{F}(S) \to \mathrm{F}(T) \times_{\mathrm{G}(T)} \mathrm{G}(S), \tag{4}$$

given by $x \mapsto (\mathrm{F}(u)(x), \phi_S(x))$ for all $x \in \mathrm{F}(S)$, is a surjection. Notice that any morphism $\phi : \mathrm{F} \to \mathrm{G}$ of functors naturally extends to a morphism $\hat{\phi} : \hat{\mathrm{F}} \to \hat{\mathrm{G}}$ of functors on $\hat{\mathbf{a}}_p$. If ϕ is a smooth morphism, then $\hat{\phi}_S : \hat{\mathrm{F}}(S) \to \hat{\mathrm{G}}(S)$ is surjective for all objects S in $\hat{\mathbf{a}}_p$.

Similarly, we say that a morphism $\phi : \mathrm{F} \to \mathrm{G}$ of functors $\mathrm{F}, \mathrm{G} : \mathbf{a}_p(n) \to \mathbf{Sets}$ on $\mathbf{a}_p(n)$ is smooth if the map of sets (4) is surjective for all surjective morphisms $u : S \to T$ in $\mathbf{a}_p(n)$. Clearly, a morphism $\phi : \mathrm{F} \to \mathrm{G}$ of functors on $\mathbf{a}_p$ is smooth if and only if the restriction $\phi_n : \mathrm{F}_n \to \mathrm{G}_n$ is smooth for all integers n.

Consider a morphism $u : S \to T$ in $\mathbf{a}_p$, and denote by $I = I(S)$ the radical of S and by $K = \ker(u)$ the kernel of u. We say that u is *small* if $IK = KI = 0$. Notice that we do not insist that K is a 1-dimensional k-vector space. If $u : S \to T$ is surjective, then u can be factored into a finite number of small surjections, since I is nilpotent. Clearly, the assignment of the map of sets (4) to the morphism u respects compositions. A morphism $\phi : \mathrm{F} \to \mathrm{G}$ of functors on $\mathbf{a}_p$ is therefore smooth if and only if the map (4) is surjective for all small surjective morphisms $u : S \to T$ in $\mathbf{a}_p$.

Let F be a fixed functor on $\mathbf{a}_p$. A *pro-couple* for F is a pair (S, ξ), where S is an object in $\hat{\mathbf{a}}_p$ and $\xi \in \hat{\mathrm{F}}(S)$ is an element. A morphism $u : (S, \xi) \to (S', \xi')$ of pro-couples is a morphism $u : S \to S'$ in $\hat{\mathbf{a}}_p$ such that $\hat{\mathrm{F}}(u)(\xi) = \xi'$. A *couple* for F is a pair (S, ξ), where S is an object of $\mathbf{a}_p$, and $\xi \in \mathrm{F}(S)$ is an element. Morphisms of couples are defined similarly. We say that a pro-couple (S, ξ) *pro-represents* F if $\alpha(\xi) : \mathrm{h}_S \to \mathrm{F}$ is an isomorphism of functors, and that a couple (S, ξ) *represents* F if $\alpha(\xi) : \mathrm{h}_S \to \mathrm{F}$ is an isomorphism of functors. It is clear that if the couple (S, ξ) represents F, then this couple is unique up to a unique isomorphism of couples.

Similarly, let F be a fixed functor on $\mathbf{a}_p(n)$. We define a couple for F to be a pair (S, ξ), where S is an object of $\mathbf{a}_p(n)$ and $\xi \in \mathrm{F}(S)$ is an element. We say that the couple (S, ξ) represents F if and only if $\alpha_n(\xi)$ is an isomorphism. It is clear that if this is the case, the couple (S, ξ) is unique up to a unique isomorphism of couples.

Let F be a functor on $\mathbf{a}_p$, and let (S, ξ) be a pro-couple for F. Then there exists a couple (S_n, ξ_n) for F_n for all integers n, given in the following way: $S_n = S/I(S)^n$, $u : S \to S_n$ is the canonical surjection, and $\xi_n = \mathrm{F}(u)(\xi)$. Notice that for all integers n, the morphism $\alpha_n(\xi_n)$ is the restriction of the morphism $\alpha(\xi)$ to $\mathbf{a}_p(n)$. It follows that (S, ξ) pro-represents F if and only if (S_n, ξ_n) represents F_n for all integers n. In particular, it follows that if there exists a pro-couple (S, ξ) which pro-represents F, then this pro-couple is unique up to a unique isomorphism of pro-couples.

Definition 5.5. *Let* F *be a functor on* $\mathbf{a}_p$*. A pro-representing hull (or a hull) for* F *is a pro-couple* (S, ξ) *such that the following conditions hold:*
i) $\alpha(\xi)$ *is a smooth morphism of functors,*
ii) (S_2, ξ_2) *represents* F_2.

We conclude this section by proving the uniqueness of the pro-representing hull, if it exists. Notice that the pro-representing hull is only unique up to *non-canonical* isomorphism.

Proposition 5.6. *Let* $\mathrm{F} : \mathbf{a}_p \to \mathbf{Sets}$ *be a functor, and assume that* $(S, \xi), (S', \xi')$ *are two pro-representing hulls for* F*. Then there exists an isomorphism of pro-couples* $u : (S, \xi) \to (S', \xi')$*.*

PROOF. Let $\phi = \alpha(\xi), \phi' = \alpha(\xi')$. Since ϕ, ϕ' are smooth morphisms, we have that $\phi_{S'}$ and ϕ'_S are surjective. So there are morphisms $u : (S, \xi) \to (S', \xi')$ and $v : (S', \xi') \to (S, \xi)$ of pro-couples for F. The restriction to $\mathbf{a}_p(2)$ gives us morphisms $u_2 : (S_2, \xi_2) \to (S'_2, \xi'_2)$ and $v_2 : (S'_2, \xi'_2) \to (S_2, \xi_2)$. But both (S_2, ξ_2) and (S'_2, ξ'_2) represent F_2, so u_2 and v_2 are inverses. In particular, $\mathrm{gr}_1(u_2)$ and $\mathrm{gr}_1(v_2)$ are inverses, and $(v \circ u)_2 = v_2 \circ u_2 = \mathrm{id}$. From the proof of proposition 5.3, we see that $\mathrm{gr}(v \circ u)$ is surjective. This means that $\mathrm{gr}_n(v \circ u)$ is a surjective endomorphims of a finite dimensional k-vector space for all $n \geq 1$, so $\mathrm{gr}(v \circ u)$ is an isomorphism. By proposition 5.3, $v \circ u$ is an isomorphism as well, and the same holds for $u \circ v$ by a symmetric argument. It follows that u and v are isomorphisms. □

4. Hulls of non-commutative deformation functors

Let R be an associative k-algebra, and $\mathbf{M} = \{M_1, \ldots, M_p\}$ a finite family of left R-modules. The aim of this section is to show that the functor $\mathrm{Def}_{\mathbf{M}}$ has a pro-representing hull (H, ξ), and furthermore to construct this hull explicitly. We will do this, following Laudal [**19**], [**20**], under the assumption that $\mathrm{Ext}^n_R(M_i, M_j)$ has finite dimension as a k-vector space for $n = 1, 2,\ 1 \leq i, j \leq p$. We remark that it is possible to generalize this result to the case of countable dimension, see Laudal [**19**].

Proposition 5.7. *Let* $u : S \to T$ *be a small, surjective morphism in* $\mathbf{a}_p$ *with kernel* $K = \ker(u)$*, and let* M_T *in* $\mathrm{Def}_{\mathbf{M}}(T)$ *be a deformation. Then there exists a canonical obstruction*

$$\mathrm{o}(u, M_T) \in (\mathrm{Ext}^2_R(M_i, M_j) \otimes_k K_{ji}),$$

such that $\mathrm{o}(u, M_T) = 0$ *if and only if there exists a deformation* M_S *in* $\mathrm{Def}_{\mathbf{M}}(S)$ *lifting* M_T*. If this is the case, the set of deformations in* $\mathrm{Def}_{\mathbf{M}}(S)$ *lifting* M_T *is a torsor under the* k*-vector space*

$$(\mathrm{Ext}^1_R(M_i, M_j) \otimes_k K_{ji}).$$

PROOF. We recall from section 2 that up to equivalence, we may assume that M_T has the following form: $M_T = (M_i \otimes_k T_{ij})$ with the natural right T-module structure, and with a left R-module structure given by the k-linear homomorphisms $r : M_i \to \oplus(M_j \otimes_k T_{ji})$ for all $r \in R$. Via the natural projections, the map r give rise to k-linear maps $r_{ij} : M_i \to M_j \otimes_k T_{ji}$ for $r \in R,\ 1 \leq i, j \leq p$. Since u is surjective, we may choose k-linear maps $L(r)_{ij} : M_i \to M_j \otimes_k S_{ji}$ such that $(id \otimes u) \circ L(r)_{ij} = r_{ij}$ for $r \in R,\ 1 \leq i, j \leq p$. Let $L(r) = (L(r)_{ij}) \in (\mathrm{Hom}_k(M_i, M_j \otimes_k S_{ji}))$, this defines a k-linear left action of R on $M_S = (M_i \otimes_k S_{ij})$, lifting the left R-module structure on M_T. We let $Q' = (\mathrm{Hom}_k(M_i, M_j \otimes_k S_{ji}))$, and remark that this is an associative k-algebra in a natural way: We compose the k-linear morphisms in Q' by using the multiplication in S.

For $r, s \in R$, consider the expression $L(rs) - L(r)L(s) \in Q'$. By the associativity of the left R-module structure on M_T, we see that $L(rs) - L(r)L(s) \in Q$, where $Q = (\mathrm{Hom}_k(M_i, M_j \otimes_k K_{ji})) \subseteq Q'$. Furthermore, we notice that $Q \subseteq Q'$ is an ideal, and therefore Q has a natural structure as a R-R bimodule via L. We define $\psi \in \mathrm{Hom}_k(R \otimes_k R, Q)$ to be given by $\psi(r, s) = L(rs) - L(r)L(s)$ for all $r, s \in R$. A straight-forward calculation shows that ψ is a 2-cocycle in $\mathrm{HC}^*(R, Q)$, so ψ gives rise to an element $\mathrm{o}(u, M_T) \in \mathrm{HH}^2(R, Q)$. (See appendix A for a general reference of Hochschild cohomology). Since u is a small morphism, $K^2 = 0$ and therefore $Q^2 = 0$. If follows that if L' is another k-linear lifting of the left R-module structure on M_T, then the R-R bimodule structures of Q given by L and L' coincide. Therefore, $\mathrm{HH}^*(R,Q)$ is independent upon the choice of L, and a straight-forward calculation shows that the same holds for the obstruction $\mathrm{o}(u,M_T)$.

We remark that there exists a deformation $M_S \in \mathrm{Def}_{\mathbf{M}}(S)$ lifting M_T if and only if there exists some k-linear lifting $L' : R \to Q'$ of the left R-module structure of M_T such that $L'(rs) = L'(r)L'(s)$ for all $r, s \in R$. Let $\tau = L' - L$, then $\tau : R \to Q$ is a k-linear map. And a straight-forward calculation shows that $L'(rs) = L'(r)L'(s)$ if and only if the relation

$$L(rs) - L(r)L(s) = L(r)\tau(s) - \tau(rs) + \tau(r)L(s) + \tau(r)\tau(s)$$

holds. Since ϕ is small, $Q^2 = 0$, and the last term vanishes. The fact that the above relation holds for all $r, s \in R$ is therefore equivalent to the fact that $\mathrm{o}(u,M_T) = 0$ in $\mathrm{HH}^2(R,Q)$. We have therefore established that there exists a canonical obstruction $\mathrm{o}(u,M_T) \in \mathrm{HH}^2(R,Q)$ such that $\mathrm{o}(u,M_T) = 0$ if and only if there is a lifting of M_T to S.

Assume that $L : R \to Q'$ is such that $L(rs) = L(r)L(s)$ for all $r, s \in R$, that is, such that it defines a deformation M_S lying over M_T. For any other k-linear lifting $L' : R \to Q'$ of the left R-module structure on M_T, we may consider the difference $\tau = L' - L : R \to Q$. A straight-forward calculation shows that τ is a 1-cocycle in $\mathrm{HC}^*(R,Q)$ if and only if $L'(rs) = L'(r)L'(s)$ for all $r, s \in R$, that is, if and only if L' defines a left R-module structure on M_S. Furthermore, we have that L and L' give rise to equivalent deformations if and only if τ is a 1-coboundary: It is clear that any equivalence between the left R-module structures of $M_S = (M_i \otimes_k S_{ij})$ given by L and L' has the form $id + \psi$, where $\psi \in Q$. Furthermore, the map $id + \psi : M_S \to M_S$ (with the left R-module structure from L' and L respectively) is a left R-module homomorphism if and only if $L(r)(id + \psi) = (id + \psi)L'(r)$ holds for all $r \in R$, and this last condition is equivalent with the fact that $\tau = d(\psi)$, so that τ is a 1-coboundary. If τ is a 1-boundary in $\mathrm{HC}^*(R,Q)$, it is also clear that $id + \psi$ defines an equivalence between the two deformations given by L and L'. Therefore, the set of deformations M_S lying over M_T is a torsor under the k-vector space $\mathrm{HH}^1(R,Q)$.

To end the proof, we have to show that there are isomorphisms of k-vector spaces $\mathrm{HH}^n(R,Q) \cong (\mathrm{Ext}^n_R(M_i,M_j) \otimes_k K_{ji})$ for $n = 1, 2$: Since $L(r)$ is a lifting to M_S of the left multiplication of r on M_T (satisfying equation 3), $L(r)$ satisfies equation 3 as well. That is, we have $L(r)_{ij}(m_i) - \delta_{ij}(rm_i) \otimes e_i \in M_j \otimes_k I_{ji}$ for all $r \in R$, $m_i \in M_i$, $1 \le i, j \le p$. But $I(S)K = 0$ since u is small, so the R-R bimodule structure of Q defined above via L coincides with the following natural one: Let $Q_{ij} = \mathrm{Hom}_k(M_i, M_j \otimes_k K_{ji})$ be the (i,j)'th component of Q, which has a natural R-R bimodule structure since M_i and $M_j \otimes_k K_{ji}$ are left R-modules in a natural way (see appendix A). This defines a component-wise bimodule structure on Q. The natural isomorphism $\mathrm{HC}^n(R,Q) \to (\mathrm{HC}^n(R,Q_{ij}))$, which is given by the natural projections, defines an isomorphism of complexes. Therefore, we have $\mathrm{HH}^n(R,Q) \cong (\mathrm{HH}^n(R,Q_{ij}))$ for all $n \ge 0$. By appendix A, proposition A.3, we have $\mathrm{HH}^n(R,Q_{ij}) \cong \mathrm{Ext}^n_R(M_i, M_j \otimes_k K_{ji})$ for $n \ge 0$. But we have $\mathrm{Ext}^n_R(M_i, M_j \otimes_k K_{ji}) \cong \mathrm{Ext}^n_R(M_i,M_j) \otimes_k K_{ji}$, since K_{ji} is a k-vector space of finite dimension. This completes the proof of proposition 5.7. $\square$

Let $u : S \to T$ and $u' : S' \to T'$ be two small surjections in $\mathbf{a}_p$, let $K = \ker(u)$ and $K' = \ker(u')$. Assume that $v : S \to S'$ and $w : T \to T'$ are morphisms such that $u' \circ v = w \circ u$. Then $v(K) \subseteq K'$, and the map v induces a k-linear map of obstruction spaces

$$(\mathrm{Ext}^2_R(M_j,M_i) \otimes_k K_{ij}) \to (\mathrm{Ext}^2_R(M_j,M_i) \otimes_k K'_{ij}).$$

From the proof of proposition 5.7, we observe that if M_T is a deformation of $\mathbf{M}$ to T and $M_{T'} = \mathrm{Def}_{\mathbf{M}}(w)(M_T)$ is the corresponding deformation to T', then this map of obstruction spaces maps $o(u, M_T)$ to $o(u', M_{T'})$.

We shall now start the construction of the pro-representing hull (H, ξ) of $\mathrm{Def}_{\mathbf{M}}$, using the obstruction theory for $\mathrm{Def}_{\mathbf{M}}$. Let us therefore fix the following notation: Let $\{s_{ij}(l) : 1 \leq l \leq d_{ij}\}$ be a basis for $\mathrm{Ext}^1_R(M_j, M_i)^*$ and $\{t_{ij}(l) : 1 \leq l \leq r_{ij}\}$ be a basis for $\mathrm{Ext}^2_R(M_j, M_i)^*$ for $1 \leq i, j \leq p$, with $d_{ij} = \dim_k \mathrm{Ext}^1_R(M_j, M_i)$ and $r_{ij} = \dim_k \mathrm{Ext}^2_R(M_j, M_i)$. Furthermore, we let $\psi^l_{ij} \in \mathrm{Der}_k(R, \mathrm{Hom}_k(M_j, M_i))$ be a representative of $s_{ij}(l)^* \in \mathrm{Ext}^1_R(M_j, M_i)$ via Hochschild cohomology. Finally, we define $\mathrm{T}^1 = \hat{S}(\{\mathrm{Ext}^1_R(M_j, M_i)^*\})$ and $\mathrm{T}^2 = \hat{S}(\{\mathrm{Ext}^2_R(M_j, M_i)^*\})$ to be the corresponding formal matrix rings in $\hat{\mathbf{a}}_p$.

First, let us show that $\mathrm{Def}_{\mathbf{M}}$ restricted to $\mathbf{a}_p(2)$ is representable: We define H_2 to be the object $H_2 = \mathrm{T}^1_2 = \mathrm{T}^1/I(\mathrm{T}^1)^2$ in $\mathbf{a}_p(2)$. For all objects S in $\mathbf{a}_p(2)$, we get $\mathrm{Mor}(H_2, S) \cong (\mathrm{Hom}_k(\mathrm{Ext}^1_R(M_j, M_i)^*, I(S)_{ij})) \cong (\mathrm{Ext}^1_R(M_j, M_i) \otimes_k I(S)_{ij})$, and $\mathrm{Def}_{\mathbf{M}}(S) \cong (\mathrm{Ext}^1_R(M_j, M_i) \otimes_k I(S)_{ij})$ from proposition 5.7 applied to the morphism $S \to k^p$. The isomorphisms we obtain in this way are compatible, so they induce an isomorphism $\phi : \mathrm{h}_{H_2} \to \mathrm{Def}_{\mathbf{M}}$ of functors on $\mathbf{a}_p(2)$. From the version of lemma 5.4 for the category $\mathbf{a}_p(2)$, we see that there is a unique deformation $\xi_2 \in \mathrm{Def}_{\mathbf{M}}(H_2)$ such that $\alpha_2(\xi_2) = \phi$. By definition, (H_2, ξ_2) represents $\mathrm{Def}_{\mathbf{M}}$ restricted to $\mathbf{a}_p(2)$.

Let us also give an explicit description of the deformation ξ_2: We have $H_2 = \mathrm{T}^1_2$, so let us denote by $\epsilon_{ij}(l)$ the image of $s_{ij}(l)$ in H_2 for $1 \leq i, j \leq p,\ 1 \leq l \leq d_{ij}$. Then ξ_2 is represented by $(M_i \otimes_k (H_2)_{ij})$ with left R-module structure given by

$$r(m_i \otimes e_i) = rm_i \otimes e_i + \sum_{j,l} \psi^l_{ji}(r)(m_i) \otimes \epsilon_{ji}(l)$$

for all $r \in R,\ m_i \in M_i$ and $1 \leq i \leq p$.

Theorem 5.8. *Let R be an associative k-algebra, and $\mathbf{M} = \{M_1, \ldots, M_p\}$ a finite family of left R-modules such that $\mathrm{Ext}^n_R(M_i, M_j)$ is a finite dimensional k-vector space for $n = 1, 2,\ 1 \leq i, j \leq p$. Then there exists a morphism $o : \mathrm{T}^2 \to \mathrm{T}^1$ in $\hat{\mathbf{a}}_p$ such that $H(\mathbf{M}) = \mathrm{T}^1 \hat{\otimes}_{\mathrm{T}^2} k^p$ is a pro-representing hull for $\mathrm{Def}_{\mathbf{M}}$.*

Proof. For simplicity, we write I for the ideal $I = I(\mathrm{T}^1)$. For all $n \geq 1$, we denote by T^1_n the quotient $\mathrm{T}^1_n = \mathrm{T}^1/I^n$, and by $t_n : \mathrm{T}^1_{n+1} \to \mathrm{T}^1_n$ the natural morphism. From the paragraphs preceding this theorem, we know that (H_2, ξ_2) represents $\mathrm{Def}_{\mathbf{M}}$ restricted to $\mathbf{a}_p(2)$. Let $o_2 : \mathrm{T}^2 \to \mathrm{T}^1_2$ be the trivial morphism (that is, o_2 is such that $o_2(I(\mathrm{T}^2)) = 0$), then $H_2 \cong \mathrm{T}^1_2 \otimes_{\mathrm{T}^2} k^p$. Using o_2 and ξ_2 as a starting point, we construct o_n for $n \geq 3$ by an inductive process. So let $n \geq 2$, and assume that the morphism $o_n : \mathrm{T}^2 \to \mathrm{T}^1_n$ is given. Furthermore, let the deformation $\xi_n \in \mathrm{Def}_{\mathbf{M}}(H_n)$ be given, with $H_n = \mathrm{T}^1_n \otimes_{\mathrm{T}^2} k^p$. Notice that for each $n \geq 2$, we may assume that o_n is constructed such that $t_{n-1} \circ o_n = o_{n-1}$, and that ξ_n is chosen as a lifting of ξ_{n-1}.

Let us now construct the morphism $o_{n+1} : \mathrm{T}^2 \to \mathrm{T}^1_{n+1}$: We let a'_n be the ideal in T^1_n generated by $o_n(I(\mathrm{T}^2))$. Then $a'_n = a_n/I^n$ for an ideal $a_n \subseteq \mathrm{T}^1$ with $I^n \subseteq a_n$, and $H_n \cong \mathrm{T}^1/a_n$. We obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{T}^2 \arrow[dr, "o_n"] & \mathrm{T}^1_{n+1} \arrow[r] \arrow[d] & \mathrm{T}^1/Ia_n \arrow[d] \\
& \mathrm{T}^1_n \arrow[r] & H_n = \mathrm{T}^1/a_n
\end{tikzcd}$$

Observe that right vertical morphism is a small surjection. So by proposition 5.7, there is an obstruction $o'_{n+1} = o(\mathrm{T}^1/Ia_n \to H_n, \xi_n)$ for lifting ξ_n to T^1/Ia_n, and we have

$$o'_{n+1} \in (\mathrm{Ext}^2_R(M_j, M_i) \otimes_k (a_n/Ia_n)_{ij}) \cong (\mathrm{Hom}_k(\mathrm{gr}_1(\mathrm{T}^2)_{ij}, (a_n/Ia_n)_{ij})).$$

Consequently, we obtain a morphism $o'_{n+1} : \mathrm{T}^2 \to \mathrm{T}^1/Ia_n$. Let a''_{n+1} be the ideal in T^1/Ia_n generated by $o'_{n+1}(I(\mathrm{T}^2))$. Then $a''_{n+1} = a_{n+1}/Ia_n$ for an ideal $a_{n+1} \subseteq \mathrm{T}^1$ with $Ia_n \subseteq a_{n+1} \subseteq a_n$. We define H_{n+1} to be $H_{n+1} = \mathrm{T}^1/a_{n+1}$, and we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{T}^2 \arrow[rr, bend left, "o'_{n+1}"] \arrow[dr, "o_n"'] & \mathrm{T}^1_{n+1} \arrow[r] \arrow[d] & \mathrm{T}^1/Ia_n \arrow[r] \arrow[d] & H_{n+1} = \mathrm{T}^1/a_{n+1} \arrow[dl] \\
& \mathrm{T}^1_n \arrow[r] & H_n = \mathrm{T}^1/a_n &
\end{tikzcd}$$

By the choice of a_{n+1}, the obstruction for lifting ξ_n to H_{n+1} is zero. We can therefore choose a deformation $\xi_{n+1} \in \mathrm{Def}_{\mathbf{M}}(H_{n+1})$ lying over ξ_n.

We know that $t_{n-1} \circ o_n = o_{n-1}$, which means that $a_{n-1} = I^{n-1} + a_n$. For simplicity, let us write $O(K) = (\mathrm{Hom}_k(\mathrm{gr}_1(\mathrm{T}^2)_{ij}, K_{ij}))$ for any ideal K, and consider the following commutative diagram of k-vector spaces, in which the columns are exact:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
O(Ia_n/I^{n+1}) \arrow[r, "t_n"] \arrow[d] & O(Ia_{n-1}/I^n) \arrow[d] \\
O(a_n/I^{n+1}) \arrow[r, "k_n"] \arrow[d, "r_{n+1}"] & O(a_{n-1}/I^n) \arrow[d, "r_n"] \\
O(a_n/Ia_n) \arrow[r, "l_n"] \arrow[d] & O(a_{n-1}/Ia_{n-1}) \arrow[d] \\
0 & 0
\end{tikzcd}$$

We may consider consider o_n as an element in $O(a_{n-1}/I^n)$. On the other hand, $o'_{n+1} \in O(a_n/Ia_n)$, and from the commutativity of the previous diagram, we have $l_n(o'_{n+1}) = r_n(o_n)$. But from the construction, o_n is an element of $O(a_n/I^n)$ as well, and we can find an element $\overline{o}_{n+1} \in O(a_n/I^{n+1})$ such that $k_n(\overline{o}_{n+1}) = o_n$. Let furthermore $\overline{o}'_{n+1} = r_{n+1}(\overline{o}_{n+1})$. Then $d' = o'_{n+1} - \overline{o}'_{n+1} \in \ker(l_n)$. But the top horizontal morphism t_n is surjective, since $a_{n-1} = a_n + I^{n-1}$. By the snake lemma, it follows that $d' = r_{n+1}(d)$ with $d \in \ker(k_n)$. We let $o_{n+1} = \overline{o}_{n+1} + d$, then $o_{n+1} \in O(a_n/I^{n+1})$ satisfies $k_n(o_{n+1}) = o_n$ and $r_{n+1}(o_{n+1}) = o'_{n+1}$. If follows that o_{n+1} defines a morphism $o_{n+1} : \mathrm{T}^2 \to \mathrm{T}^1_{n+1}$ in $\hat{\mathbf{a}}_p$ such that $t_n \circ o_{n+1} = o_n$ and such that $\mathrm{T}^1_{n+1} \otimes_{\mathrm{T}^2} k^p \cong H_{n+1}$.

Using this construction, we obtain a morphism $o_n : \mathrm{T}^2 \to \mathrm{T}^1_n$ and a deformation $\xi_n \in \mathrm{Def}_{\mathbf{M}}(H_n)$, with $H_n = \mathrm{T}^1_n \otimes_{\mathrm{T}^2} k^p$, for all $n \geq 1$. From the construction, we see that there are morphisms $h_n : H_{n+1} \to H_n$ for all $n \geq 1$, and we have $H_{n+1}/I(H_{n+1})^n \cong H_n$. Let $o : \mathrm{T}^2 \to \mathrm{T}^1$ be the morphism defined by the projective limit of the morphisms o_n, and let H be the corresponding object in $\mathbf{A}_p$,

$$H = \mathrm{T}^1 \hat{\otimes}_{\mathrm{T}^2} k^p \cong \varprojlim H_n.$$

Then, $H/I(H)^n \cong H_{n+1}/I(H_{n+1})^n \cong H_n$, so H is an object in the pro-category $\hat{\mathbf{a}}_p$. Let $\xi \in \mathrm{Def}_{\mathbf{M}}(H)$ be the deformation given by the projective limit of the deformations ξ_n. Then (H, ξ) is a pro-couple for $\mathrm{Def}_{\mathbf{M}}$. It only remains to show that (H, ξ) is a pro-representable hull for $\mathrm{Def}_{\mathbf{M}}$.

It is clearly enough to show that (H_n, ξ_n) is a pro-representing hull for $\mathrm{Def}_{\mathbf{M}}$ restricted to $\mathbf{a}_p(n)$ for all $n \geq 3$. So let $\phi = \alpha_n(\xi_n)$ be the morphism of functors on $\mathbf{a}_p(n)$ corresponding to ξ_n. We shall prove that ϕ is a smooth morphism. So let $u : S \to T$ be a small surjection in $\mathbf{a}_p(n)$, and assume that $M_S \in \mathrm{Def}_{\mathbf{M}}(S)$ and $v \in \mathrm{Mor}(H, T)$ are given such that $\mathrm{Def}_{\mathbf{M}}(u)(M_S) = \phi_T(v) = M_T$. Since $I(T)^n = 0$, we may consider v as a morphism in $\mathrm{Mor}(H_n, T)$. Now, let us consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{T}^1/Ia_n \arrow[d] & \\
\mathrm{T}^1 \arrow[ur] \arrow[r] \arrow[dr] & H_{n+1} \arrow[d] & S \arrow[d, "u"] \\
 & H_n \arrow[r, "v"'] & T
\end{tikzcd}$$

Let $v'' : \mathrm{T}^1 \to S$ be any morphism making the diagram commutative. Then $v''(a_n) \in \ker(u)$, so $v''(Ia_n) = 0$ since u is small. But the obstruction $o(u, M_T)$ is obviously zero, so $v''(a_{n+1}) = 0$, and we obtain a morphism $v'' \in \mathrm{Mor}(H_{n+1}, S)$ making the above diagram commutative. Since $v''(I(H_{n+1})^n) = 0$, we may consider v'' as a morphism from $H_{n+1}/I(H_{n+1})^n \cong H_n$, that is, $v'' \in \mathrm{Mor}(H_n, S) = \mathrm{Mor}(H, S)$. We denote by M'_S the corresponding deformation in $\mathrm{Def}_{\mathbf{M}}(S)$. Since M_S, M'_S are two liftings of M_T, the difference is an element $d \in (\mathrm{Ext}^1_R(M_j, M_i) \otimes_k K_{ij})$ by proposition 5.7, with $K = \ker(u)$. Hence, d defines a map $D : \mathrm{T}^1 \to S$ such that $D(I) \subseteq K$. Clearly, $D(I^2) = 0$ since u is small, so D factorizes through H. Let $v' \in \mathrm{Mor}(H, S)$ be given by $v'(s) = v''(s) + D(s)$ for all $s \in I(H)$. Notice that this is a ring homomorphism since u is small. Then $\phi_S(v') = M_S$ by construction, and $u \circ v' = u \circ v'' = v$ since $D(I) \subseteq K = \ker(u)$. It follows that ϕ is smooth, and this concludes the proof of the theorem. □

The theorem shows that the deformation functor $\mathrm{Def}_{\mathbf{M}}$ has a pro-representing hull $H = H(\mathbf{M})$ when the conditions of the theorem are fulfilled. In this case, we know from proposition 5.6 that this hull must be unique, up to (non-canonical) isomorphism. Furthermore, theorem 5.8 shows that the hull H can be calculated from the obstruction morphism $o : \mathrm{T}^2 \to \mathrm{T}^1$, which is defined by the obstruction theory for $\mathrm{Def}_{\mathbf{M}}$, as described in proposition 5.7.

There is another procedure for calculating the pro-representing hull H, using the cohomology groups $\mathrm{Ext}^1_R(M_i, M_j)$ and certain generalized Massey products (called matrix Massey products) defined on these cohomology groups. These Massey products are in a sense dual to the obstruction morphism $o : \mathrm{T}^2 \to \mathrm{T}^1$. In the case of the deformation functor of a module, in the sense of Schlessinger [**30**], this procedure is explained in detail in Laudal [**20**] using symmetric matrix Massey products. It is possible to adapt this procedure to our non-commutative deformation functors, using a non-symmetric variation of these matrix Massey products.

We say that a ring R is left (right) *hereditary* if the following condition holds: For all left (right) R-modules M, N, we have $\mathrm{Ext}^n_R(M, N) = 0$ for all $n \geq 2$. We say that R is *hereditary* if it is left and right hereditary. From theorem 5.8, it is clear that if R is a left hereditary ring, then $H = \mathrm{T}^1$ is a hull for the deformation functor $\mathrm{Def}_{\mathbf{M}}$ for any finite family $\mathbf{M}$ of left R-modules such that $\dim_k \mathrm{Ext}^1_R(M_i, M_j)$ is finite for $1 \leq i, j \leq p$. In this paper, we shall only be interested in non-commutative deformation functors over hereditary rings. For this reason, we choose not to go into the details of non-symmetric matrix Massey products in this paper.

5. Classification of extensions of extensions

Let R be an associative k-algebra, and let $\mathbf{M} = \{M_1, \dots, M_p\}$ be a finite family of left R-modules. Throughout this section, we shall assume that the modules $M_1, \dots, M_p$ are non-isomorphic as left R-modules.

We define an extension of extensions of the family $\mathbf{M}$ to be a couple (M, F), where M is a left R-module and F is a finite, descending filtration of M of left R-modules

$$M = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{n-1} \supseteq F_n = 0,$$

such that for $1 \leq i \leq n$, $F_{i-1}/F_i \cong M_{l_i}$ with $1 \leq l_1, \dots, l_n \leq p$. We say that the couple (M, F) has *length* n and *order vector* $\underline{l} = (l_1, \dots, l_n)$. Notice that the order vector is uniquely defined by the filtration F, since the modules $M_1, \dots, M_p$ are non-isomorphic.

We denote by a *co-filtration* of M of length n a chain of surjections of left R-modules

$$M = C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 = 0.$$

This notion is equivalent with the notion of an ascending filtration on M of length n: If F is a filtration, let C be the co-filtration with $C_i = M/F_i$ and such that the surjection $C_i \to C_{i-1}$ is induced by $\mathrm{id} : M \to M$. Conversely, if C is a co-filtration of M, let F be the filtration given by $F_i = \ker(M \to C_i)$ and the natural inclusions. We observe that $\ker(C_i \to C_{i-1}) = \operatorname{coker}(F_i \to F_{i-1}) = F_{i-1}/F_i$ for $1 \leq i \leq n$.

It turns out that it is most useful to represent an extension of extensions of the family $\mathbf{M}$ as a couple (M, C), where M is a left R-module and C is a co-filtration of M. We shall therefore keep this notation, and write n for the length of the co-filtration and $\underline{l}$ for the order vector. Notice that we have short exact sequences

$$0 \to M_{l_i} \to C_i \to C_{i-1} \to 0$$

for $1 \leq i \leq n$. So C_i is an extension of C_{i-1} with M_{l_i} for $1 \leq i \leq n$, and in particular M is an extension of extensions of the family $\mathbf{M}$ in the sense of chapter 3. This justifies the name.

Let (M, C) and (M', C') be two extensions of extensions of the family $\mathbf{M}$. We say that (M, C) and (M', C') are *equivalent* extensions of extensions if there is an isomorphism $\phi : M \to M'$ such that $C'_i = \phi(C_i)$ for all integers i with $0 \leq i \leq n$. In particular, we see that equivalent extensions of extensions have the same length n and order vector $\underline{l}$. When ϕ defines an equivalence between (M, C) and (M', C'), we also see that for $1 \leq i \leq n$, the extension

$$0 \to M_{l_i} \to C_i \to C_{i-1} \to 0$$

corresponds to the extension

$$0 \to M_{l_i} \to C'_i \to C'_{i-1} \to 0$$

via the map $\operatorname{Ext}^1_R(C_{i-1}, M_{l_i}) \to \operatorname{Ext}^1_R(C'_{i-1}, M_{l_i})$.

Let (M, C) be an extension of extensions, and let n be the length and $\underline{l}$ be the order vector of (M, C). We define an *ordered, directed graph* to be a directed graph (N, E), where N is a set of nodes and E is a set of edges with a given total order. For each extension of extensions (M, C), we assign an ordered, directed graph $\mathrm{G} = \mathrm{G}(M, C)$ to (M, C): The graph G has nodes $N = \{1, 2, \dots, p\}$ and edges $E = \{a_1, a_2, \dots, a_{n-1}\}$, where a_i is an edge from node l_i to node l_{i+1}, and the total order on E is given by $a_1 < a_2 < \cdots < a_{n-1}$. Notice that the ordered, directed graph G is uniquely defined by the length n and the order vector $\underline{l}$ of (M, C). We call G the *extension type* of the extension of extensions (M, C). The previous remark implies that equivalent extensions of extensions have the same extension type.

Let G be an ordered, directed graph. We say that G is *finite* if G has a finite number of nodes and a finite number of edges. In this case, we may assume that the nodes of G are $N = \{1, 2, \dots, p\}$ and that the edges of G are $E = \{a_1, \dots, a_{n-1}\}$ with $a_1 < \dots < a_{n-1}$ for some natural numbers p, n. If the start of edge a_i is the same node as the end of edge a_{i-1} for $2 \leq i \leq n-1$, we say that the ordered, directed graph G is *connected*. For an ordered, directed graph which is finite and connected, we shall write l_i for the node which is the starting point of edge a_i for $1 \leq i \leq n-1$, and l_n for the node which is the ending point of arrow $n-1$.

Let G be an ordered, directed graph which is finite and connected. We define the associative k-algebra $k[\mathrm{G}]$ in the following way: For $1 \leq i \leq p$, let e_i be the idempotent $e_i = (0, \dots, 1, \dots, 0) \in k^p$. We define $k[\mathrm{G}]$ to be the associative k-algebra in $\mathbf{A}_p$ generated by k^p and $x_1, \dots, x_{n-1}$, where x_i are generators which satisfy the matrix relations $x_i = e_{l_{i+1}} x_i e_{l_i}$ for all i and the additional relations $x_i x_j = 0$ if $i \leq j$. It follows that $k[\mathrm{G}]$ satisfies $I^n = 0$, where I is the radical of $k[\mathrm{G}]$, and that $k[\mathrm{G}]$ has finite dimension as a k-vector space. Consequently, $k[\mathrm{G}]$ is an object in $\mathbf{a}_p$.

Assume that the family $\mathbf{M} = \{M_1, \dots, M_p\}$ is such that $\dim_k \mathrm{Ext}^1_R(M_i, M_j)$ is finite for $1 \leq i, j \leq p$. Then the deformation functor $\mathrm{Def}_{\mathbf{M}} : \mathbf{a}_p \to \mathbf{Sets}$ has a pro-representing hull $H = H(\mathbf{M})$ by theorem 5.8, and we know that this hull is unique up to non-canonical isomorphism. For any ordered, directed graph G which is finite and connected, we consider the set $X(\mathbf{M}, \mathrm{G}) = \mathrm{Mor}(H, k[\mathrm{G}])$ of morphisms from H to $k[\mathrm{G}]$ in the category in $\hat{\mathbf{a}}_p$.

Lemma 5.9. *Let* $\mathbf{M}$ *be a family of left* R*-modules such that* $\dim_k \mathrm{Ext}^1_R(M_i, M_j)$ *is finite for* $1 \leq i, j \leq p$*, and let* G *be an ordered, directed graph which is finite and connected. Then* $X(\mathbf{M}, \mathrm{G})$ *has a natural structure as an affine algebraic variety.*

PROOF. Let $V^h_{ij} = \mathrm{Ext}^h_R(M_j, M_i)^*$ for $1 \leq i, j \leq p$ and $h = 1, 2$. We define $\mathrm{T}^1 = \hat{S}(\{V^1_{ij}\})$ to be the formal matrix ring defined by the vector spaces V^1_{ij}, and $\mathrm{T}^2 = \hat{S}(\{V^2_{ij}\})$ to be the formal matrix ring defined by the vector spaces V^2_{ij}. Furthermore, we let $\{s_{ij}(l) : 1 \leq l \leq d_{ij}\}$ be a basis for V^1_{ij} and $\{t_{ij}(l) : 1 \leq l \leq r_{ij}\}$ be a basis for V^2_{ij} for $1 \leq i, j \leq p$. Clearly, the surjection $\mathrm{T}^1 \to H$ defines an injective map $X(\mathbf{M}, \mathrm{G}) \to \mathrm{Mor}(\mathrm{T}^1, k[\mathrm{G}]) = \mathrm{Mor}(\mathrm{T}^1_n, k[\mathrm{G}])$, where $n-1$ is the number of edges in G. Furthermore, we have that

$$\mathrm{Mor}(\mathrm{T}^1_n, k[\mathrm{G}]) \cong \prod_{i,j} \mathrm{Hom}_k(\mathrm{Ext}^1_R(M_j, M_i)^*, W_{ij}),$$

with $W = I(k[\mathrm{G}])$. This means that $\mathrm{Mor}(\mathrm{T}^1, k[\mathrm{G}]) \cong \mathbf{A}^N$, where N is given by $N = \sum (d_{ij} \ \dim_k W_{ij})$. We choose a basis $\{w_{ij}(m) : 1 \leq m \leq v_{ij}\}$ for W_{ij}. Then we obtain a set of coordinates $z_{ij}(l, m)$ for the affine space $\mathbf{A}^N$, such that each $z_{ij}(l, m)$ corresponds to a morphism $\phi_{ij}(l, m)$ given by

$$\phi_{ij}(l, m)(s_{ij}(l')) = \delta_{l,l'} w_{ij}(m)$$

for $1 \leq i, j \leq p$, $1 \leq l, l' \leq d_{ij}$, $1 \leq m \leq v_{ij}$. Consequently, we may write $\mathrm{Mor}(\mathrm{T}^1, k[\mathrm{G}]) = \mathrm{Spec}\, k[\{z_{ij}(l, m)\}]$. Let $\phi = (\alpha_{ij}(l, m))$ be a point in the affine space $\mathbf{A}^N$, and let $f_{ij}(l)$ be the image of $o(t_{ij}(l))$ in T^1_n for $1 \leq i, j \leq p$, $1 \leq l \leq r_{ij}$. We know that $\phi \in X(\mathbf{M}, \mathrm{G})$ if and only if $\phi(f_{ij}(l)) = 0$ for all $f_{ij}(l)$. But we have

$$\phi(f_{ij}(l)) = \sum_m f_{ij}(l)(\alpha_{ij}(l, m)) \ w_{ij}(m),$$

so this condition translates to the equations

$$f_{ij}(l)(\alpha_{ij}(l,m)) = 0$$

for $1 \leq i,j \leq p,\ 1 \leq l \leq d_{ij},\ 1 \leq m \leq v_{ij}$. It is clear that for each i,j,l,m, this equation induces a polynomial relation $R_{i,j}(l,m) \in k[\{z_{ij}(l,m)\}]$. It follows that $X(\mathbf{M},\mathrm{G}) = V(\{R_{ij}(l,m)\}) \subseteq \mathbf{A}^N$ is an affine, algebraic variety. □

We let $A(\mathbf{M},\mathrm{G})$ be the affine coordinate ring of the affine variety $X(\mathbf{M},\mathrm{G})$. Then we have $A(\mathbf{M},\mathrm{G}) = k[\{z_{ij}(l,m)\}]/R$, where $R \subseteq k[\{z_{ij}(l,m)\}]$ is the ideal generated by $\{R_{ij}(l,m)\}$.

Let M_H be the versal family defined over the hull H. That is, let M_H be a deformation $M_H \in \mathrm{Def}_{\mathbf{M}}(H)$ such that (H, M_H) is a pro-representing hull for $\mathrm{Def}_{\mathbf{M}}$. For any $\phi \in X(\mathbf{M},\mathrm{G})$, let M_ϕ be the deformation

$$M_\phi = \mathrm{Def}_{\mathbf{M}}(\phi)(M_H).$$

This is a deformation in $\mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$ which depends on ϕ. From the definition of the versal family M_H, the map of sets $X(\mathbf{M},\mathrm{G}) \to \mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$, given by $\phi \mapsto M_\phi$, is surjective.

Let us describe the deformation M_ϕ in detail, when $\phi \in X(\mathbf{M},\mathrm{G})$ is given: As usual, we let M_ϕ be given as the trivial right $k[\mathrm{G}]$-module

$$M_\phi = (M_i \otimes_k k[\mathrm{G}]_{ij}),$$

and let M_ϕ have a left R-module structure given in the following way: Let $m_i \in M_i$, and consider $m_i \otimes e_i$ as an element in M_H. Then $r(m_i \otimes e_i) \in M_H$, and we define left multiplication by r on the element $(m_i \otimes e_i) \in M_\phi$ to be

$$r(m_i \otimes e_i) = (1 \otimes \phi)(r(m_i \otimes e_i)),$$

where $r(m_i \otimes e_i)$ on the right side is the multiplication in $M_H = (M_i \otimes_k H_{ij})$, and $1 \otimes \phi : (M_i \otimes_k H_{ij}) \to (M_i \otimes_k k[\mathrm{G}]_{ij})$ is the map given by the equation $(1 \otimes \phi)(m_i \otimes h_{ij}) = m_i \otimes \phi(h_{ij})$ for all $m_i \in M_i,\ h_{ij} \in H_{ij}$.

Let $\tilde{M} \in \mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$ represent a deformation of the family $\mathbf{M}$ to $k[\mathrm{G}]$. Clearly, we have $\tilde{M} = (M_i \otimes_k k[\mathrm{G}]_{ij})$ considered as a right $k[\mathrm{G}]$-module. We shall denote by $M' = (M_i \otimes_k k[\mathrm{G}]_{i,l_1})$ the l_1'th column of $\tilde{M}$. We have seen that $M' \subseteq \tilde{M}$ is invariant under the left R-module action on $\tilde{M}$, so M' has a natural left R-module structure. For each ordered subsequence $\underline{j} = (j_1, \dots, j_r) \subseteq (n-1, n-2, \dots, 1)$ with $l_{j_r} = l_1$, let us write $M'(\underline{j}) = M_{l_{j_1+1}} \otimes x_{j_1} x_{j_2} \dots x_{j_r} \subseteq M'$. We shall write $M'(U)$ for the sum of $M'(\underline{j})$ for all sequences j of the form $(i, i-1, \dots, 2, 1)$ for some integer i with $0 \leq i \leq n-1$, and we shall write $M'(B)$ for the sum of $M'(\underline{j})$ for all other subsequences j with $l_{j_r} = l_1$. Then $M'(U), M'(B) \subseteq M'$ are k-linear subspaces with $M'(U) + M'(B) = M'$. Moreover, a calculation shows that $M'(B)$ is invariant under the left R-action on M'. So we have an R-linear homomorphism $M' \to M'/M'(B)$. We shall denote by M the underlying left R-module of $M'/M'(B)$, and we see that $M \cong M'(U)$ considered as k-vector spaces.

Let us construct a co-filtration of the left R-module M in the following way: Let $F_j \subseteq M$ be given by

$$F_j = \sum_{i=j}^{n-1} M_{l_{i+1}} \otimes x_i x_{i-1} \dots x_1$$

for $0 \leq j \leq n-1$. We see that F_j is invariant under the left R-action for all j. So we a have a co-filtration of M of left R-modules of the form

$$M = C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 = 0$$

with $C_i = M/F_i$ for $0 \leq i \leq n-1$ and $C_n = M$. It is clear that the co-filtration C is such that $\ker(C_i \to C_{i-1}) \cong M_{l_i}$ as left R-modules for $1 \leq i \leq n$. So (M, C) is an extension of extensions of the family $\mathbf{M}$ of extension type G.

We also see that if $\tilde{M}, \tilde{N}$ are two equivalent deformations of $\mathbf{M}$ to $k[\mathrm{G}]$, then the corresponding isomorphism

$$\phi : \tilde{M} \to \tilde{N}$$

of R-$k[\mathrm{G}]$ bimodules has the property that $\phi(M') \subseteq N'$ and $\phi(M'(B)) \subseteq N'(B)$. So ϕ induces an isomorphism $\phi : M \to N$ of left R-modules. If C, C' are the corresponding filtrations of M' and N' respectively, then it is furthermore clear that $\phi(C_i) \subseteq C'_i$ for $0 \leq i \leq n$.

Proposition 5.10. *Let $\mathbf{M} = \{M_1, \dots, M_p\}$ be a family of left R-modules such that $\dim_k \operatorname{Ext}^1_R(M_i, M_j)$ is finite for $1 \leq i, j \leq p$, and let* G *be an ordered, directed graph which is finite and connected. Then there is a natural bijective map between $\mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$ and the set of equivalence classes of extensions of extensions of the family $\mathbf{M}$ with extension type* G*.*

Proof. The construction given above clearly defines a map of sets from $\mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$ to the set of equivalence classes of extensions of extensions of $\mathbf{M}$ with extension type G. We construct an inverse of this map: Let (M, C) be an extension of extensions of $\mathbf{M}$ with extension type G. We let $\tilde{M} = (M_i \otimes_k k[\mathrm{G}]_{ij})$ considered as a right $k[\mathrm{G}]$-module. Then it is easy to see that the left R-module structure of M and the filtration C of M generate a unique left R-module structure on $\tilde{M}$, such that $\tilde{M}$ represents a deformation in $\mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$. Furthermore, equivalent extensions of extensions of $\mathbf{M}$ with extension type G give equivalent deformations in $\mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$. $\square$

We define an equivalence relation on the set $X(\mathbf{M}, \mathrm{G})$ in the following way: For morphisms $\phi, \phi' \in X(\mathbf{M}, \mathrm{G})$, we say that ϕ, ϕ' are equivalent if and only if M_ϕ and $M_{\phi'}$ are equivalent deformations in $\mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$. That is, $\phi \sim \phi'$ if and only if $M_\phi = M_{\phi'}$ in $\mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$. From the proposition above, we see that ϕ, ϕ' are equivalent if and only if the corresponding extensions of extensions (M, C) and (M', C') are equivalent extensions of extensions.

Consider the set of left R-modules M which has a co-filtration C such that (M, C) is an extension of extensions of the family $\mathbf{M}$ with extension type G. We denote by $E(\mathbf{M}, \mathrm{G})$ the set of isomorphism classes of left R-modules of this type. Clearly, we have a surjective map of sets from the set of equivalence classes of extensions of extensions of the family $\mathbf{M}$ with extension type G to $E(\mathbf{M}, \mathrm{G})$, given by $(M, C) \mapsto M$. So we have a sequence of surjections

$$X(\mathbf{M}, \mathrm{G}) \to \mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}]) \to E(\mathbf{M}, \mathrm{G}).$$

Let S be any commutative k-algebra. We define an *algebraic family of deformations of* $\mathbf{M}$ *to* $k[\mathrm{G}]$ *parametrized by* S to be an R-$(k[\mathrm{G}] \otimes_k S)$ bimodule $M(\mathbf{M}, \mathrm{G}, S)$ such that the following conditions hold:

i) $M(\mathbf{M}, \mathrm{G}, S) = (M_i \otimes_k k[\mathrm{G}]_{ij}) \otimes_k S$ considered as right S-module.
ii) For each closed point $P \in \operatorname{Spec} S$, the corresponding fiber $M(\mathbf{M}, \mathrm{G}, S)_P$, given by the R-$k[\mathrm{G}]$ bimodule $M(\mathbf{M}, \mathrm{G}, S)_P = M(\mathbf{M}, \mathrm{G}, S) \otimes_S \mathbf{k}(P)$, is a deformation of $\mathbf{M}$ to $k[\mathrm{G}]$.

Using this notation, we obtain the following proposition:

Proposition 5.11. *Let $\mathbf{M} = \{M_1, \dots, M_p\}$ be a family of left R-modules such that $\dim_k \operatorname{Ext}^1_R(M_i, M_j)$ is finite for $1 \leq i, j \leq p$, and let* G *be an ordered, directed graph which is finite and connected. Then there exists a family $M(\mathbf{M}, \mathrm{G})$ of deformations of $\mathbf{M}$ to $k[\mathrm{G}]$ parametrized by $A(\mathbf{M}, \mathrm{G})$ such that the fiber $M(\mathbf{M}, \mathrm{G})_\phi$ is the deformation $M_\phi \in \mathrm{Def}_{\mathbf{M}}(k[\mathrm{G}])$ for all closed points $\phi \in X(\mathbf{M}, \mathrm{G})$.*

PROOF. Let $M(\mathbf{M}, \mathrm{G}) = (M_i \otimes_k k[\mathrm{G}]_{ij}) \otimes_k A(\mathbf{M}, \mathrm{G})$ with trivial structure as right $k[\mathrm{G}] \otimes_k A(\mathbf{M}, \mathrm{G})$-module, and with left R-module structure (modulo the squares of the radicals) given by

$$\begin{aligned} r((m_i \otimes e_i) \otimes 1) &= ((rm_i \otimes e_i) \otimes 1) \\ &\quad + \sum_{j,l,m} (1 \otimes \phi_{ji}(l,m))(r(m_i \otimes e_i) - rm_i \otimes e_i) \otimes z_{ji}(l,m), \end{aligned}$$

where $r(m_i \otimes e_i) - rm_i \otimes e_i$ is considered as an element in M_H, and we write $z_{ji}(l,m)$ for the image of $z_{ji}(l,m)$ in $A(\mathbf{M}, \mathrm{G})$. It is now straight-forward to check that this family has the desired properties. □

Corollary 5.12. *Let* $\mathbf{M} = \{M_1, \dots, M_p\}$ *be a family of left* R*-modules such that* $\dim_k \mathrm{Ext}^1_R(M_i, M_j)$ *is finite for* $1 \leq i, j \leq p$*, and let* G *be an ordered, directed graph which is finite and connected. Then there exists a family of left* R*-modules parametrized over* $A(\mathbf{M}, \mathrm{G})$*, realized as a quotient family of the family* $M(\mathbf{M}, \mathrm{G})$*, such that all equivalence classes of* $E(\mathbf{M}, \mathrm{G})$ *occur as fibers of this family. In particular,* $E(\mathbf{M}, \mathrm{G})$ *is the quotient of the affine algebraic variety* $X(\mathbf{M}, \mathrm{G})$ *defined by this family.*

CHAPTER 6

Graded holonomic $\mathbf{A}_1(k)$-modules

Throughout this chapter, we fix the numerical semigroup $\Gamma = \mathbf{N_0}$. We shall therefore write $A = k[\Gamma] = k[t]$ and $D = \mathrm{D}(A) = A_1(k)$. The purpose of this chapter is to classify the indecomposable, graded, holonomic D-modules over the Weyl algebra $D = A_1(k)$. We do this in two steps: First, we continue the work from chapter 3 in order to classify all simple, graded D-modules in the case $\Gamma = \mathbf{N_0}$. Secondly, we use the methods from chapter 5 to classify all indecomposable D-modules which are extensions of extensions of the simple, graded D-modules.

We observe that the indecomposable, graded, holonomic D-modules correspond exactly to the indecomposable, holonomic modules with regular singularities in the local, analytic case, see Briançon and Maisonobe [**7**] and Boutet de Monvel [**6**] for the classification in this case.

1. Simple, graded D-modules

Let $\Gamma = \mathbf{N_0}$, let $A = k[\Gamma] = k[t]$, and let $D = \mathrm{D}(A) = \mathrm{A}_1(k)$. Let furthermore M be a simple, graded D-module. Then, we recall from section 3.7 that M is either an S-torsion D-module or an S-torsion free D-module. In the first case, $S^{-1}M = 0$, and we shall study this case in more detail.

The maximal ideals of A are exactly the ideals $(t-\alpha) \subseteq A$ for all $\alpha \in k$. Given an element $\alpha \in k$, we denote by $\mathbf{k}(\alpha)$ the residue field $\mathbf{k}(\alpha) = A/(t-\alpha)$. This is an A-module, and hence $V(\alpha) = D \otimes_A \mathbf{k}(\alpha)$ is a well-defined D-module for all $\alpha \in k$. It is easy to see that D is a free, right A-module with basis $\{\partial^i : i \geq 0\}$. This means that any element $p \in V(\alpha)$ can be written uniquely as a polynomial $p = p(\partial) \in k[\partial]$. Furthermore, if we identify $V(\alpha)$ with $k[\partial]$ and write $*$ for the induced left multiplication by D, we see that $t * p = \alpha p - p'$, and $\partial * p = \partial p$ for all $p \in k[\partial]$, where p' denotes the formal derivative of p.

Theorem 6.1. *Let $D = \mathrm{A}_1(k)$. Then the set* $\mathrm{Simple}(D)[S-torsion]$ *of isomorphism classes of simple, S-torsion D-modules is given by $\{V(\alpha) : \alpha \in k\}$. Furthermore, $V(\alpha) \cong D/D(t-\alpha)$ for all $\alpha \in k$.*

Proof. See Block [**4**], proposition 4.1 and corollary 4.1. □

For $\alpha = 0$, we obtain the simple, graded, torsion D-module $V(0) = D/Dt$. A simple argument shows that this is the only $V(\alpha)$ with $\alpha \in k$ which admits a graded structure. In fact, we have the following result:

Corollary 6.2. *Let $D = \mathrm{A}_1(k)$. Then the set* $\text{gr-Simple}(D)[S-torsion]$ *of equivalence classes of simple, graded S-torsion D-modules is $\{V(0)\}$.*

Proof. Let M be a simple, graded, S-torsion D-module. Then clearly, there is an isomorphism $M \cong V(\alpha)$ of D-modules for some unique $\alpha \in k$. Assume that $\alpha \neq 0$. Since M is a simple, graded D-module, there is an isomorphism $M \cong D/I$ of graded D-modules for some maximal, homogeneous left ideal $I \subseteq D$ (up to a twist). So D/I and $D/D(t-\alpha)$ are isomorphic D-modules, and the isomorphism $\phi : D/Dt \to D/I$ is given by right multiplication with some $Q \in D$ in the sense that $\phi(\overline{P}) = \overline{PQ}$ for all $P \in D$. Write $Q = Q_1 + \cdots + Q_n$, where Q_i is non-zero and homogeneous for all i and $\deg(Q_1) < \deg(Q_2) < \cdots < \deg(Q_n)$. If $Q_1 \in I$, right multiplication with Q and right multiplication with $(Q - Q_1)$ induce identical maps. We may therefore assume that $Q_1 \notin I$. But since $(t - \alpha)Q \in I$, we obtain $-\alpha Q_1 + tQ_1 - \alpha Q_2 + \cdots + tQ_n \in I$. This gives $-\alpha Q_1 \in I$, since this is the homogeneous part of degree deg(Q?1) of $(t - \alpha)Q$ and I is homogeneous. Consequently, $Q_1 \in I$ and this is a contradiction, so it follows that $\alpha = 0$. Finally, we have from lemma 3.25 that there exists a unique graded structure on $V(0)$, up to graded isomorphisms of degree 0 and twist, and this ends the proof. □

The second case is that of simple, graded D-modules that are S-torsion free. From proposition 3.26, we know that there is a bijective correspondence between the set gr-Simple(D)[$S-$torsion free] of equivalence classes of simple, graded, S-torsion free D-modules, and the set gr-Simple(D(T)) of equivalence classes of simple, graded D(T)-modules. Furthermore, this bijective correspondence is given by $M \mapsto S^{-1}M$.

Let $I \subseteq k$ be a subset of k containing 0, such that the natural, set-theoretic map $I \to k/\mathbf{Z}$ is a bijection. From corollary 3.28, we know that the set gr-Simple(D(T)) is given by $\{N_\alpha : \alpha \in I\}$, where $N_\alpha = \mathrm{D}(T)/\mathrm{D}(T)(E - \alpha)$ for all $\alpha \in I$. We also see that $N_0 = \mathrm{D}(T)/\partial\,\mathrm{D}(T) \cong T$ considered as a D(T)-module.

For each $\alpha \in I^* = I \setminus \{0\}$, let us consider the D-module $M_\alpha = D/D(E - \alpha)$. Since $I^* \subseteq k \setminus \mathbf{Z}$, it follows from Dixmier [**11**], lemma 24 that M_α is a simple D-module for all $\alpha \in I^*$. Furthermore, let us consider the D-module $M_0 = D/D\partial \cong A$. It is clear that M_0 is a simple D-module, see corollary 2.9. Consequently, we have that M_α is a simple, graded D-module with $S^{-1}M_\alpha = N_\alpha$ for all $\alpha \in I$.

Corollary 6.3. *Let $D = \mathrm{A}_1(k)$. Then the set* gr-Simple(D) *of equivalence classes of simple, graded D-modules is given by* $\{M_\alpha : \alpha \in I\} \cup \{V(0)\}$.

2. Graded holonomic D-modules

Let $A = k[t]$ and let $D = \mathrm{A}_1(k)$. We recall that a D-module M is holonomic if $M = 0$ or $M \neq 0$ and $d(M) = 1$. This means that M is holonomic if and only if M is an Artinian D-module. In particular, any holonomic D-module has a composition series

$$M = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{n-1} \supseteq F_n = 0,$$

where $n = l(M)$ it the length of M, and F_i/F_{i-1} is a simple D-module for $1 \leq i \leq n$. Equivalently, M has a co-filtration of the form

$$M = C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 = 0,$$

where the homomorphism $C_i \to C_{i-1}$ is surjective with simple kernel for $1 \leq i \leq n$.

Assume that M is a graded holonomic D-module. Then M is Noetherian and Artinian, so M satisfies the ACC and DCC for submodules of M. We say that the graded module M is gr-Noetherian (gr-Artinian) if the set of homogeneous submodules of M satisfies the ACC (DCC). In this notation, the graded holonomic D-module M is clearly gr-Noetherian and gr-Artinian, since it is graded, Noetherian and Artinian. This means that M has a co-filtration

$$M = C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 = 0,$$

such that $C_i \to C_{i-1}$ is a graded surjection of degree 0 with gr-simple kernel for $1 \leq i \leq n$. This is clear, since each homogeneous ideal $I \subseteq D$ with $I \neq D$ is either maximal among the homogeneous ideals, or there is a minimal element among the homogeneous ideals J with $I \subseteq J$ and $I \neq J$. From proposition 3.24, any gr-simple D-module is a simple, graded D-module. So in particular, n is the length of the holonomic D-module M, and the filtration corresponding to C is a graded composition series for M.

Proposition 6.4. *Let $D = \mathrm{A}_1(k)$ and let M be a non-zero, graded, holonomic D-module of length n. Then there exist a co-filtration of M of the form*

$$M = C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 = 0,$$

where $C_i \to C_{i-1}$ is a graded surjection of degree 0 with a simple, graded kernel for $1 \leq i \leq n$.

3. Extensions of extensions of simple, graded D-modules

Consider the set gr-Simple$(D) = \{M_\alpha : \alpha \in I\} \cup \{V(0)\}$ of equivalence classes of graded, simple D-modules. It is essential to calculate the extensions between the simple, graded modules in order to classify the extensions of extensions of simple, graded D-modules. It will be useful to write M_∞ for $V(0)$ and $\hat{I}$ for $I \cup \{\infty\}$, so we shall use this notation.

Lemma 6.5. *Let $\alpha, \beta \in \hat{I}$. If $\alpha = \beta \in I^*$ or if $\alpha = 0$ and $\beta = \infty$ or if $\alpha = \infty$ and $\beta = 0$, then $\dim_k \mathrm{Ext}^1_D(M_\alpha, M_\beta) = 1$. In all other cases, $\mathrm{Ext}^1_D(M_\alpha, M_\beta) = 0$.*

Proof. This is a straight-forward calculation. □

For $\alpha \in I^*$, we have a k-linear basis $\{t^n : n \geq 0\} \cup \{\partial^n : n > 0\}$ for M_α. Furthermore, we have a k-linear basis $\{t^n : n \geq 0\}$ for M_0 and $\{\partial^n : n \geq 0\}$ for M_∞. In all cases where $\dim_k \mathrm{Ext}^1_D(M_\alpha, M_\beta) = 1$, we shall give a derivation $\psi \in \mathrm{Der}_k(D, \mathrm{Hom}_k(M_\alpha, M_\beta))$ such that the equivalence class of ψ is a k-linear basis for $\mathrm{Ext}^1_D(M_\alpha, M_\beta)$. Since t, ∂ generate D as a k-algebra, it is enough to give the endomorphisms ψ_t and ψ_∂.

Assume that $\alpha = \beta \in I^*$. Then we choose ψ such that $\psi_t(t^n) = 0,\ \psi_\partial(\partial^n) = 0$ for all $n \geq 0$, and such that $\psi_t(\partial^n) = \partial^{n-1},\ \psi_\partial(t^n) = t^{n-1}$ for $n > 0$. If $\alpha = 0$ and $\beta = \infty$, we choose ψ such that $\psi_t = 0$ and $\psi_\partial(1) = 1,\ \psi_\partial(t^n) = 0$ for $n > 0$. If $\alpha = \infty$ and $\beta = 0$, we choose ψ such that $\psi_\partial = 0$ and $\psi_t(1) = 1,\ \psi_t(\partial^n) = 0$ for $n > 0$.

Let R be any associative, $\mathbf{Z}$-graded k-algebra, and let M, N be $\mathbf{Z}$-graded R-modules. If R is a Noetherian ring and M is an R-module of finite type, then $\mathrm{Ext}^n_R(M, N)$ is a $\mathbf{Z}$-graded k-vector space for all $n \geq 0$. Let $\psi \in \mathrm{Der}_k(\mathrm{Hom}_k(M, N))$ represent $\xi \in \mathrm{Ext}^1_R(M, N)$ via Hochschild cohomology. Then ξ is homogeneous of degree w if and only if $\psi(R_i)(M_j) \subseteq M_{i+j+w}$ for all $i, j \in \mathbf{Z}$.

Let R be an associative, $\mathbf{Z}$-graded k-algebra, and let M, N be $\mathbf{Z}$-graded R-modules. A *graded extension* of M with N of degree d is a short exact sequence

$$0 \to N \to E \to M \to 0$$

of graded R-modules, where $f : N \to E$ is a graded homomorphism of degree d and $g : E \to M$ is a graded homomorphism of degree 0. A pair of graded extensions (E, f, g), (E', f', g') of degree d are equivalent if there exists a graded isomorphism $\tau : E \to E'$ of degree 0, such that $\tau f = f'$ and $g = g'\tau$. We denote by $\mathrm{Ext}(M, N)_d$ the set of equivalence classes of graded extensions of M with N of degree d. We refer to appendix A for comparison with the well-known, non-graded case version.

Proposition 6.6. *Let R be an associative, $\mathbf{Z}$-graded k-algebra, and let M, N be $\mathbf{Z}$-graded left R-modules. Then there is a bijective correspondence between $\mathrm{Ext}(M, N)_d$ and $\mathrm{Ext}^1_R(M, N)_{-d}$.*

Proof. From the non-graded case, we know that E is an extension of M with N if and only if $E = N \oplus M$ considered as k-vector space, with R-module structure given by $r(n, m) = (rn + \psi_r(m), rm)$ for some $\psi \in \mathrm{Der}_k(R, \mathrm{Hom}_k(M, N))$. Notice that E has a graded structure compatible with $N \to E \to M$ if and only if it is homogeneous of degree $-d$ in the sense that $\psi(R_i)(M_j) \subseteq M_{i+j-d}$ for all integers i, j. We know that the two extensions E, E' are equivalent via $\tau : E \to E'$ if and only if there is a co-boundary ϕ such that $\tau(n, m) = (n + \phi(m), m)$. But if $d\tau = \psi - \psi'$ with ψ, ψ' homogeneous of degree $-d$, then τ is homogeneous of degree $-d$ as well, in the sense that $\phi(M_i) \subseteq M_{i-d}$ for all integers i. This proves the claim. □

Let $\alpha, \beta \in \hat{I}$. From the calculations of $\operatorname{Ext}^1_D(M_\alpha, M_\beta)$ given above, we see that any extension $\xi \in \operatorname{Ext}^1_D(M_\alpha, M_\beta)$ satisfies $\xi = 0$ or ξ is homogeneous of some degree d. Moreover, the latter case only occurs for the following values for α and β:
i) $\alpha = \beta \in I^*$ with $d = 0$.
ii) $\alpha = 0$ and $\beta = \infty$ with $d = 1$.
iii) $\alpha = \infty$ and $\beta = 0$ with $d = -1$.
Consequently, any extension of M_α with M_β is graded. Moreover, the extension is either split or a graded extension of degree $-d$, and the latter case only occurs for the values of α and β mentioned above.

Let $\mathbf{M} = \{M_1, \dots, M_p\}$ be a finite family of graded D-modules. A *graded extension of extensions* of the family $\mathbf{M}$ is a couple (M, F) consisting of a graded D-module M and a filtration C of M of the form

$$M = C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 = 0,$$

where $C_i \to C_{i-1}$ is a graded surjection of degree 0 for $1 \leq i \leq n$, and there is a graded isomorphism $\ker(C_i \to C_{i-1}) \to M_{l_i}$ of degree d_i for some integers d_i, l_i with $1 \leq l_i \leq p$.

Corollary 6.7. *Let M be a non-zero, graded, holonomic D-module of length n. Then there exists a finite subset $I(M) \subseteq I$ and a co-filtration C of M of length n, such that (M, C) is a graded extension of extensions of $\{M_\alpha : \alpha \in I(M)\}$. Moreover, every extension of extensions of a finite subfamily of $\{M_\alpha : \alpha \in \hat{I}\}$ of length n is a non-zero, graded, holonomic D-module.*

Proof. The first part follows from proposition 6.4 and the fact that any simple, graded D-module is equivalent to M_α for some $\alpha \in \hat{I}$, up to graded isomorphisms of degree 0 and twists. For the second part, it is enough to recall that any extension of M_α with M_β is graded for all $\alpha, \beta \in \hat{I}$. □

It follows from the corollary that in order to classify the graded, holonomic D-modules it is enough to classify all extensions of extensions of sub-families of the family $\{M_\alpha : \alpha \in \hat{I}\}$ of simple, graded D-modules.

4. Indecomposable, graded D-modules

We say that a D-module M is *decomposable* if $M \cong N_1 \oplus N_2$ for some D-submodules $N_1, N_2 \subseteq M$ with $N_1, N_2 \neq 0$. We say that M is *indecomposable* if it is not decomposable. In particular, all simple D-modules are indecomposable. It is well-known that every D-module is a direct sum of indecomposable D-modules, since D is a Noetherian ring (recall that we only consider D-modules of finite type).

It is therefore of interest to classify the indecomposable D-modules. We shall do this via the theory of extension of extensions, and the following lemma is useful:

Lemma 6.8. *Let R be any ring, and let $0 \to N \to E \to M \to 0$ be a short exact sequence of R-modules. If M is decomposable, then E is decomposable as well.*

Proof. Assume that we have morphisms $f : N \to E$ and $g : E \to M_1 \oplus M_2$ such that $0 \to N \to E \to M_1 \oplus M_2 \to 0$ is exact. Let $E_i = g^{-1}(M_i)$ for $i = 1, 2$. Then

E_1, E_2 are submodules of E such that $E_1 + E_2 = E$ and $E_1 \cap E_2 = 0$. So the result follows. □

This lemma implies that if (M, C) is an extension of extensions, then M is indecomposable if and only if C_i is indecomposable for all integers i with $0 \leq i \leq n$. This observation has the following consequence:

Proposition 6.9. *Let $D = \mathrm{A}_1(k)$ and let (M, C) be an extension of extensions of a finite sub-family of the simple, graded D-modules. If M is an indecomposable D-module, then the short exact sequence*

$$0 \to \ker(C_i \to C_{i-1}) \to \ker(C_i \to C_{i-2}) \to \ker(C_{i-1} \to C_{i-2}) \to 0$$

is non-split for $2 \leq i \leq n$.

Proof. Let us denote the extension corresponding to the above exact sequence by ξ_i for $2 \leq i \leq n$. It is immediately clear that $\xi_2 \neq 0$ from the lemma above. So let $m > 2$, and assume that $\xi_i \neq 0$ for all $i < m$. If $\xi_m = 0$, then it is clear that there exists a split extension ξ' of C_{m-1} with $K_m = \ker(C_m \to C_{m-1})$ compatible with the extension ξ_m. So the difference between ξ' and the given extension

$$0 \to K_m \to C_m \to C_{m-1} \to 0$$

is given by an extension of C_{m-2} to K_m. We see that the simple components of C_{m-1} are either all isomorphic to M_α for some fixed $\alpha \in I^*$ (case I), or alternately isomorphic to M_0 and M_∞ (case II): This follows since $\xi \neq 0$ for all $i < m$. We also see that if $\mathrm{Ext}^1_D(N, K_m) = 0$ for each simple component N of C_{m-2}, then $\mathrm{Ext}^1(C_{m-2}, K_m) = 0$ as well. But then ξ_m is a sum of split extensions, and therefore split. This is a contradiction, so in case I, we must have $K_m \cong M_\alpha$ and in case II we must have $K_m \cong M_0$ or $K_m \cong M_\infty$. In either case, we see that K_m is isomorphic to a simple component of C_{m-1}. But then we may add the extension of C_{m-2} with K_m to the D-module C_{m-1} to obtain a new D-module C'_{m-1}, in such a way that $C_m = K_m \oplus C'_{m-1}$. But this is a contradiction, since C_m is indecomposable. So $\xi_i \neq 0$ for all i by induction on m. □

The proposition implies that if M is an indecomposable, graded D-module, then is an extension of extensions of one of the following types:

- Type I: $\mathbf{M} = \{M_1\}$ with $M_1 = M_\alpha$ for some $\alpha \in I^*$, and G is given by the arrows $a_1, \dots, a_{n-1}$ from node 1 to node 1 with $a_1 < a_2 < \dots < a_{n-1}$.
- Type II.A: $\mathbf{M} = \{M_1, M_2\}$ with $M_1 = M_0$ and $M_2 = M_\infty$, and G is given by the arrows $a_1, \dots, a_{n-1}$ with $a_1 < \dots < a_{n-1}$, where a_i is an arrow from node 1 to node 2 for all odd numbers i, and an arrow from node 2 to node 1 for all even numbers i.
- Type II.B: $\mathbf{M} = \{M_1, M_2\}$ with $M_1 = M_0$ and $M_2 = M_\infty$, and G is given by the arrows $a_1, \dots, a_{n-1}$ with $a_1 < \dots < a_{n-1}$, where a_i is an arrow from node 2 to node 1 for all odd numbers i, and a_i is an arrow from node 1 to node 2 for all even numbers i.

Furthermore, we have seen that given an indecomposable, graded D-module M, then any co-filtration C on M such that (M, C) is an extension of extensions of simple, graded D-modules has the property that the short exact sequence

$$0 \to \ker(C_i \to C_{i-1}) \to \ker(C_i \to C_{i-2}) \to \ker(C_{i-1} \to C_{i-2}) \to 0$$

is non-split for $2 \leq i \leq n$.

5. Extensions of extensions of type I

Let the family $\mathbf{M}$ be given by $\mathbf{M} = \{M_1\}$, with $M_1 = M_\alpha$ for some $\alpha \in I^*$. Let furthermore G be the ordered, directed graph with a single node $N = \{1\}$, and with edges $E = \{a_1, \dots, a_{n-1}\}$ for some $n \geq 1$, where a_i is an edge from node 1 to node 1 for $1 \leq i \leq n-1$ and the total order of E is given by $a_1 < \cdots < a_{n-1}$. We want to apply methods from chapter 5 to classify all indecomposable D-modules in $E(\mathbf{M}, \mathrm{G})$.

The k-algebra $H = \mathrm{T}^1 = k[[s_{11}]]$ is a pro-representing hull of the deformation functor $\mathrm{Def}_{\mathbf{M}} : \mathbf{a}_1 \to \mathbf{Sets}$. This is clear, since D is a hereditary ring. We have to find the versal family M_H over H: We know that $M_H = M \hat{\otimes}_k H$ considered as right H-module. Furthermore, we have that

$$P(m \otimes 1) = Pm \otimes 1 + \psi_P(m) \otimes s_{11} + \Delta(P, m)$$

for all $P \in D$, $m \in M_1$, where ψ is given as above and $\Delta(P, m) \in M_H I(H)^2$. It is easy to see that $\Delta(t, m) = \Delta(\partial, m) = 0$ defines a versal deformation M_H if and only if $\psi_\partial \psi_t - \psi_t \psi_\partial = 0$, since $[\partial, t] = 1$ is the only relation in D. But $\psi_\partial \psi_t = \psi_t \psi_\partial = 0$ in $\mathrm{End}_k(M_1)$, so it follows that the left R-module structure on M_H is given by

$$P(m \otimes 1) = Pm \otimes 1 + \psi_P(m) \otimes s_{11}$$

for $P = t, \partial$ and $m \in M_1$.

Let $k[\mathrm{G}]$ be the k-algebra of the ordered, directed graph G. We see that we have $k[\mathrm{G}] = k[x_1, \dots, x_{n-1}]$, where x_i are generators with the relations $x_i x_j = 0$ if $i \geq j$. However, we shall consider the k-algebra $k[\mathrm{G}'] = k[x]/(x^n)$ in $\mathbf{a}_1$ and the affine, algebraic variety $X(\mathbf{M}, \mathrm{G}') = \mathrm{Mor}(H, k[\mathrm{G}']) \subseteq \mathrm{Mor}(H, k[\mathrm{G}])$. These objects are obtained when we consider the ordered, directed graph G' with relations $a_i = a_{i-1}$ for $1 \leq i \leq n-1$. Let $A(\mathbf{M}, \mathrm{G}')$ be the affine coordinate ring of $X(\mathbf{M}, \mathrm{G}') = \mathbf{A}^{n-1}$. Then $A(\mathbf{M}, \mathrm{G}') = k[z_1, \dots, z_{n-1}]$, where z_i corresponds to the the morphism $\phi_i : H \to k[\mathrm{G}']$ given by $\phi_i(s_{11}) = x^i$ for $1 \leq i \leq n-1$. We construct a family $M(\mathbf{M}, \mathrm{G}')$ of deformations of $\mathbf{M}$ to $k[\mathrm{G}']$ parametrized by $A(\mathbf{M}, \mathrm{G}')$, in a manner similar to the construction of the family $M(\mathbf{M}, \mathrm{G})$ for G.

We observe that the family $M(\mathbf{M}, \mathrm{G}')$ is versal in the sense that it contains all indecomposable isomorphism classes of $E(\mathbf{M}, \mathrm{G})$ as fibers: It is clear that it contains all indecomposable isomorphism classes of $E(\mathbf{M}, \mathrm{G}')$ as fibers. But we have that $\dim_k \mathrm{Ext}^1_D(M_\alpha, M_\alpha) = 1$, and from proposition 6.9, the extensions

$$0 \to \ker(C_i \to C_{i-1}) \to \ker(C_i \to C_{i-2}) \to \ker(C_{i-1} \to C_{i-2}) \to 0$$

are non-split for $2 \leq i \leq n$, since M is indecomposable. So the indecomposable isomorphism classes of $E(\mathbf{M}, \mathrm{G})$ and $E(\mathbf{M}, \mathrm{G}')$ coincide.

Let us proceed to find the versal family $M(\mathbf{M}, \mathrm{G}')$ of deformations of $\mathbf{M}$ to $k[\mathrm{G}']$ parametrized by $A(\mathbf{M}, \mathrm{G}')$: Clearly, we have

$$M(\mathbf{M}, \mathrm{G}') = (M_1 \otimes_k k[\mathrm{G}']) \otimes_k A(\mathbf{M}, \mathrm{G}'),$$

considered as a right $k[\mathrm{G}'] \otimes_k A(\mathbf{M}, \mathrm{G}')$-module. Furthermore, we see that the left D-module structure is given by the formula

$$P(m \otimes 1 \otimes 1) = (Pm) \otimes 1 \otimes 1 + \sum_{i=1}^{n-1} \psi_P(m) \otimes x^i \otimes z_i$$

for $P = t, \partial$ and $m \in M_1$. More explicitly, this gives

$$
\begin{aligned}
P(m \otimes x^{n-1} \otimes 1) &= (Pm) \otimes x^{n-1} \otimes 1 \\
P(m \otimes x^{n-2} \otimes 1) &= (Pm) \otimes x^{n-2} \otimes 1 + \psi_P(m) \otimes x^{n-1} \otimes z_1 \\
P(m \otimes x^{n-3} \otimes 1) &= (Pm) \otimes x^{n-3} \otimes 1 + \psi_P(m) \otimes x^{n-2} \otimes z_1 + \psi_P(m) \otimes x^{n-1} \otimes z_2 \\
&\vdots \\
P(m \otimes x \otimes 1) &= (Pm) \otimes x \otimes 1 + \sum_{i=1}^{n-2} \psi_P(m) \otimes x^{i+1} \otimes z_i \\
P(m \otimes 1 \otimes 1) &= (Pm) \otimes 1 \otimes 1 + \sum_{i=1}^{n-1} \psi_P(m) \otimes x^{i} \otimes z_i
\end{aligned}
$$

for $P = t, \partial$ and $m \in M_1$.

Let $m_1 = 1 \otimes 1 \otimes 1 \in M(\mathbf{M}, \mathrm{G}')$. Then we see from the equations above that $M(\mathbf{M}, \mathrm{G}')$ is generated by m_1 considered as a D-$k[\mathrm{G}'] \otimes_k A(\mathbf{M}, \mathrm{G}')$ bimodule. Let us write $\tilde{D} = D \otimes_k (k[\mathrm{G}'] \otimes_k A(\mathbf{M}, \mathrm{G}'))^{op}$. Then we may consider $M(\mathbf{M}, \mathrm{G}')$ as a left $\tilde{D}$-module with the following free resolution:

$$0 \leftarrow M(\mathbf{M}, \mathrm{G}') \leftarrow \tilde{D} \xleftarrow{C} \tilde{D} \leftarrow 0,$$

where C denotes right multiplication with the element $C \in \tilde{D}$ given by

$$C = (E - \alpha) \otimes (1 \otimes 1)^{op} - 1 \otimes (\sum_{i=1}^{n-1} x^i \otimes z_i)^{op}.$$

We know that there is a Kodaira-Spencer map for the family $M(\mathbf{M}, \mathrm{G}')$, which we denote

$$g : \mathrm{Der}_k(A(\mathbf{M}, \mathrm{G}') \to \mathrm{Ext}^1_{\tilde{D}}(M(\mathbf{M}, \mathrm{G}'), M(\mathbf{M}, \mathrm{G}')).$$

We write $\mathbf{V}$ for the kernel of this map, and it is well-known that the maximal integral manifolds of $X(\mathbf{M}, \mathrm{G}') = \mathbf{A}^{n-1}$ with respect to $\mathbf{V}$ consist of points over which the fibers are isomorphic considered as D-$k[\mathrm{G}']$ bimodules. A computation of the Kodaira-Spencer kernel $\mathbf{V}$ in this case gives the k-vector space

$$\mathbf{V} = < z_1\partial_1 + \cdots + z_{n-1}\partial_{n-1}, z_1\partial_2 + \cdots + z_{n-2}\partial_{n-1}, \ldots, z_1\partial_{n-1} >,$$

where $\partial_i = \partial/\partial z_i$ for $1 \leq i \leq n-1$. In particular, we see that all points in $X(\mathbf{M}, \mathrm{G}')$ with $z_1 \neq 0$ have fibers which are isomorphic considered as D-$k[\mathrm{G}']$ bimodules.

Corollary 6.10. *Let $D = \mathrm{A}_1(k)$. For each integer $n \geq 1$, there is a unique graded, holonomic, indecomposable D-module M which is an extension of extensions of type I, up to graded isomorphism of degree 0 and twists. Moreover, $M = D/D(E - \alpha)^n$ is a representative of this equivalence class.*

Proof. From the computation of the Kodaira-Spencer kernel $\mathbf{V}$ above, we see that there can be at most one equivalence class of the above mentioned type for each $n \geq 1$. But it is clear that the D-module $M = D/D(E-\alpha)^n$ is a graded, holonomic D-module for all n. Furthermore, we have an exact sequence

$$0 \to D(E - \alpha)^{i-1}/D(E - \alpha)^i \to D/D(E - \alpha)^i \to D/D(E - \alpha)^{i-1} \to 0$$

for $1 \leq i \leq n$, and $D(E - \alpha)^{i-1}/D(E - \alpha)^i \cong M_\alpha$ for all i. So it follows that M is an extension of extensions of type I. Finally, a direct computation shows that $\mathrm{End}_D(M) = k[E - \alpha]/(E - \alpha)^n$ is a local ring, so M is an indecomposable D-module. $\square$

6. Extensions of extensions of type II

Let the family $\mathbf{M}$ be given by $\mathbf{M} = \{M_1, M_2\}$, with $M_1 = M_0$ and $M_2 = M_\infty$. Let furthermore G be the ordered, directed graph with nodes $N = \{1, 2\}$, and with edges $E = \{a_1, \dots, a_{n-1}\}$ for some $n \geq 1$, with $a_1 < \dots < a_{n-1}$. For type II.A, we have that a_i is an edge from node 1 to node 2 for all odd integers i with $1 \leq i \leq n-1$ and an edge from node 2 to node 1 for all even integers i with $1 \leq i \leq n-1$. For type II.B, we have that a_i is an edge from node 2 to node 1 for all odd integers i with $1 \leq i \leq n-1$ and an edge from node 1 to node 2 for all even integers i with $1 \leq i \leq n-1$.

Assume that $n = 2m$ is even. Then $M_A = D/D(t\partial)^m$ and $M_B = D/D(\partial t)^m$ are graded, holonomic D-modules which are extensions of extensions of type II.A and II.B respectively. Similarly, if $n = 2m+1$ is odd, then $M_A = D/D\partial(t\partial)^m$ and $M_B = D/Dt(\partial t)^m$ are graded, holonomic D-modules which are extensions of extensions of type II.A and II.B respectively.

Lemma 6.11. *Let $n \geq 1$ be a positive integer, and let M_A, M_B be the graded, holonomic D-modules given above. Then $\mathrm{End}_D(M_A), \mathrm{End}_D(M_B)$ are local rings. In particular, M_A and M_B are indecomposable D-modules.*

Proof. A direct computation shows that if $n = 2m$ is even, then the endomorphism rings are $\mathrm{End}_D(M_A) = k[E]/E^m$ and $\mathrm{End}_D(M_B) = k[E+1]/(E+1)^m$. Similarly, the endomorphism rings are given by $\mathrm{End}_D(M_A) = k[E]/E^{m+1}$ and $\mathrm{End}_D(M_B) = k[E+1]/(E+1)^{m+1}$ when $n = 2m+1$ is odd. Since all these rings are commutative, local k-algebras, the result follows. □

It turns out that it is difficult to calculate the subset of indecomposable isomorphism classes in $E(\mathbf{M}, \mathrm{G})$ for type II extension of extensions using the techniques from section 5. We shall therefore use a more direct computation to obtain this subset. The key to the new approach, is the following computation of Ext-groups:

Lemma 6.12. *We have $\dim_k \mathrm{Ext}^1_D(M_A, M_0) = \dim_k \mathrm{Ext}^1_D(M_B, M_\infty) = 1$ when $n = 2m$ is even, and we have $\dim_k \mathrm{Ext}^1_D(M_A, M_\infty) = \dim_k \mathrm{Ext}^1_D(M_B, M_0) = 1$ when $n = 2m+1$ is odd.*

Proof. This follows from a straight-forward calculation. □

Corollary 6.13. *Let $D = \mathrm{A}_1(k)$. For each integer $n \geq 1$, M_A and M_B are the unique graded, holonomic, indecomposable D-modules which are extensions of extensions of type II.A and type II.B, up to graded isomorphism of degree 0 and twists.*

Proof. It is clear that the D-modules M_A and M_B given above are graded, holonomic and indecomposable D-modules which are extensions of extensions of type II.A and type II.B. Let M be one of these modules, and let C be a co-filtration of type II. Then C_i is an extension of C_{i-1} with $K_i = \ker(C_i \to C_{i-1})$ for all $i \geq 2$. But from lemma 6.12, we have $\mathrm{Ext}^1_D(K_i, C_{i-1}) \cong k$, and since M is indecomposable, C_i is not a trivial extension. But this means that C_i is the unique possible extension of C_{i-1} with K_i, for all $i \geq 2$, up to isomorphism of D-modules. By induction, the uniqueness of M_A and M_B follows. □

CHAPTER 7

Graded holonomic D-modules

In chapter 6, we have classified all the graded, holonomic, indecomposable modules over the Weyl-algebra $D = \mathrm{A}_1(k)$. We recall that we have fixed some subset $I \subseteq k$ such that I contains 0 and the natural map $I \to k/\mathbf{Z}$ is a bijection of sets. Furthermore, we write $I^* = I \setminus \{0\}$. Using this notation, we may summarize the results from chapter 6 in the following theorem:

Theorem 7.1. *Let $D = \mathrm{A}_1(k)$ and let $n \geq 1$ be a positive integer. The set of graded, holonomic, indecomposable D-modules of length n, up to graded isomorphisms of degree 0 and twists, are given by*

$$\{D/D(E-\alpha)^n : \alpha \in I^*\} \cup \{D/DE^m,\ D/D(\partial t)^m\}$$

when $n = 2m$ is even, and by

$$\{D/D(E-\alpha)^n : \alpha \in I^*\} \cup \{D/D\partial E^m,\ D/Dt(\partial t)^m\}$$

when $n = 2m+1$ is odd.

Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let $D = \mathrm{D}(A)$ be the ring of differential operators on A. From corollary 2.9, we know that D is Morita equivalent with the Weyl algebra $\mathrm{A}_1(k)$. Furthermore, it is clear that the Morita equivalence preserves decompositions of modules, so the property of being indecomposable is a Morita equivalent property.

Let us denote by $M(n,\alpha)$ the D-module corresponding to $\overline{D}/\overline{D}(E-\alpha)^n$ for all $\alpha \in I^*$, $n \geq 1$. Let us furthermore denote by $M_A(n)$ the D-module corresponding to $\overline{D}/\overline{D}E^m$ when $n = 2m$ is even and to $\overline{D}/\overline{D}\partial E^m$ when $n = 2m+1$ is odd, and by $M_B(n)$ the D-module corresponding to $\overline{D}/\overline{D}(\partial t)^m$ when $n = 2m$ is even and to $\overline{D}/\overline{D}t(\partial t)^m$ when $n = 2m+1$ is odd.

Theorem 7.2. *Let Γ be a numerical semigroup, let $A = k[\Gamma]$ be the corresponding monomial curve, and let $D = \mathrm{D}(A)$ be the ring of differential operators on A. The set of graded, holonomic, indecomposable D-modules of length n, up to graded isomorphisms of degree 0 and twists, are given by*

$$\{M(\alpha, n) : \alpha \in I^*\} \cup \{M_A(n), M_B(n)\}.$$

APPENDIX A

The Ext groups and Hochschild cohomology

Let R be an associative k-algebra, and let M, N be left R-modules. In this appendix, we recall several different descriptions of the k-vector space $\mathrm{Ext}^n_R(M, N)$ for $n \geq 0$. In particular, we discuss the description given by Hochschild cohomology of R with values in the R-R bimodule $\mathrm{Hom}_k(M, N)$. We also define the cup product on Ext groups.

1. The Yoneda description of Ext groups

Let M, N be left R-modules, and fix free resolutions (L_*, d_*) of M and (L'_*, d'_*) of N. We shall write the differentials $d_i : L_{i+1} \to L_i$ and $d'_i : L'_{i+1} \to L'_i$. Furthermore, we denote the augmentation morphisms by $\rho : L_0 \to M$ and $\rho' : L'_0 \to N$.

For integers $n \geq 0$, $\mathrm{Ext}^n_R(M, N)$ is defined to be the n'th cohomology group of the complex $Hom_R(L_*, N)$,

$$\mathrm{Ext}^n_R(M, N) = H^n(Hom_R(L, N)).$$

Notice that in general, this Abelian group does not have a left R-module structure, but only a left $\mathrm{C}(R)$-module structure, where $\mathrm{C}(R)$ is the centre of R. In particular, if R is commutative, then $\mathrm{Ext}^n_R(M, N)$ has the structure of an R-module for $n \geq 0$, and if R is a k-algebra, then $\mathrm{Ext}_R(M)$ has the structure of a k-vector space.

We denote by $\mathrm{Hom}^*(L_*, L'_*)$ the *Yoneda complex*, given in the following way: For each integer $n \geq 0$, let $\mathrm{Hom}^n(L_*, L'_*)$ be the left R-module

$$\mathrm{Hom}^n(L_*, L'_*) = \amalg_i \mathrm{Hom}_R(L_{i+n}, L'_i),$$

where Hom_R denotes the left R-linear maps. We write $\phi = (\phi_i)$ for an element $\phi \in \mathrm{Hom}^n(L_*, L'_*)$, where $\phi_i \in \mathrm{Hom}_R(L_{i+n}, L'_i)$ for all $i \geq 0$, and we define the map $d^n : \mathrm{Hom}^n(L_*, L'_*) \to \mathrm{Hom}^{n+1}(L_*, L'_*)$ by the formula

$$d^n(\phi)_i = \phi_i d_{n+i} + (-1)^{n+1} d'_i \phi_{i+1}$$

for $i \geq 0$. It is then easy to check that this map is a well-defined differential, and it is a morphism of Abelian groups. We shall write $H^n(\mathrm{Hom}(L_*, L'_*))$ for the cohomology groups of the Yoneda complex. Since the differential $d = d^n$ is left $\mathrm{C}(R)$-linear, these cohomology groups have a natural structure as left $\mathrm{C}(R)$-modules.

Lemma A.1. *For all integers $n \geq 0$, we have a canonical isomorphism*

$$H^n(\mathrm{Hom}(L_*, L'_*)) \cong \mathrm{Ext}^n_R(M, N)$$

Proof. There is a natural map $f_n : \mathrm{Hom}^n(L_*, L'_*) \to \mathrm{Hom}_R(L_n, N)$, given by $f(\phi) = \rho' \phi_0$, where $\phi = (\phi_i) \in \mathrm{Hom}^n(L_*, L'_*)$. It is easy to see that these maps are compatible with the differentials, and a small calculation show that f_n induces an isomorphism on cohomology $H^n(\mathrm{Hom}(L_*, L'_*)) \to \mathrm{Ext}^n_R(M, N)$ for all integers $n \geq 0$. $\square$

Let $f : L \to L'$ be a left R-linear map between finite, free left R-modules. Then, we shall always represent this map f as right multiplication by a matrix F over R. This is done by choosing bases for L, L', and by letting the i'th row of F

represent the image of the i'th basis vector of L under f. Notice that if $g : L' \to L''$ is another map of the same kind, then gf is represented by the matrix FG.

2. Definition of Hochschild cohomology

Let R be an associative k-algebra, and let Q be an R-R bimodule. We define $\mathrm{HC}^n(R,Q) = \mathrm{Hom}_k(\otimes_k^n R, Q)$ for all $n \geq 0$. To simplify notation, we shall write $\psi(r_1,\dots,r_n)$ in place of $\psi(r_1 \otimes \dots \otimes r_n)$ for $\psi \in \mathrm{HC}^n(R,Q)$, $r_1,\dots,r_n \in R$. Furthermore, we define maps $d^n : \mathrm{HC}^n(R,Q) \to \mathrm{HC}^{n+1}(R,Q)$ by the formula

$$(5) \quad d^n(\psi)(r_0,\dots,r_n) = r_0\,\psi(r_1,\dots,r_n) + \sum_{i=1}^{n} (-1)^i \psi(r_0,\dots,r_{i-1}r_i,\dots,r_n) + (-1)^{n+1}\psi(r_0,\dots,r_{n-1})\,r_n$$

for all $\psi \in \mathrm{HC}^n(R,Q)$, $r_0,\dots,r_n \in R$. It is clear that the maps d^n are k-linear maps.

Lemma A.2. $\mathrm{HC}^*(R,Q)$ *is a complex of k-vector spaces.*

Proof. Let $\psi \in \mathrm{HC}^n(R,Q)$. Then $\psi' = d^n(\psi)$ is a sum of $n+1$ summands, and we denote these by $\psi'_0,\dots,\psi'_n$, in the order they appear in formula 5. Let $\psi'' = d^{n+1}\psi' = d^{n+1}d^n\psi$. Each $d^{n+1}\psi'_i$ for $0 \leq i \leq n$ is a sum of $n+2$ summands, and we denote these by ψ''_{ij} for $0 \leq j \leq n+1$ in the order they appear in formula 5. A straight-forward calculation shows that we have $\psi''_{i,j} + \psi''_{j,i+1} = 0$ for all indices i,j with $0 \leq j \leq n+2$, $j \leq i \leq n+1$. Since $\psi'' = \sum \psi''_{ij}$, it follows that $\psi'' = 0$ in $\mathrm{HC}^{n+2}(R,Q)$. Consequently, $\mathrm{HC}^*(R,Q)$ is a complex of k-vector spaces. □

We denote the corresponding cohomology group $\mathrm{HH}^n(R,Q)$ the n'th Hochschild cohomology group of R with values in Q. Explicitly, we define $\mathrm{HH}^n(R,Q)$ to be given by

$$\mathrm{HH}^n(R,Q) = H^n(\mathrm{HC}^*(R,Q)) = \ker(d^n)/\operatorname{Im}(d^{n-1})$$

for all $n \geq 0$. In particular, the cohomology groups $\mathrm{HH}^n(R,Q)$ have a natural structure as k-vector spaces.

We also notice that an element $\psi \in \mathrm{HC}^1(R,Q)$ is a 1-cocycle if and only if $\psi(rs) = r\psi(s) + \psi(r)s$ for all $r,s \in R$. So we have $\ker(d^1) = \mathrm{Der}_k(R,Q)$. We say that a derivation $\psi \in \mathrm{Der}_k(R,Q)$ is trivial if there is a $q \in Q$ such that ψ is of the form $\psi(r) = rq - qr$ for all $r \in R$. Clearly, the set of trivial derivations is the image $\operatorname{Im}(d^0)$.

3. Hochschild cohomology in terms of Ext groups

Let M,N be left R-modules. Then $Q = \mathrm{Hom}_k(M,N)$ is an R-R bimodule in a natural way: For any $r \in R$, let $L_r : M \to M$ denote left multiplication on M by r, and $L'_r : N \to N$ denote left multiplication on N by r. The bimodule structure is given by $r\phi = L'_r\phi$, $\phi r = \phi L_r$ for $r \in R$, $\phi \in \mathrm{Hom}_k(M,N)$. We shall consider the Hochschild cohomology of R with values in $Q = \mathrm{Hom}_k(M,N)$.

By definition, we have that $\mathrm{HH}^0(R,Q) = \ker(d^0)$, and furthermore it is clear that $\ker(d^0) = \mathrm{Hom}_R(M,N)$ when $Q = \mathrm{Hom}_k(M,N)$. So we have a natural isomorphism of k-vector spaces $\mathrm{Ext}^0_R(M,N) \cong \mathrm{HH}^0(R,Q)$. Notice that since $k \subseteq \mathrm{C}(R)$, $\mathrm{Ext}^n_R(M,N)$ has a natural k-vector space structure for all $n \geq 0$. We remark that it is possible to extend the above isomorphism to the higher cohomology groups:

Proposition A.3. *Let R be an associative k-algebra, M,N be left R-modules and $\mathrm{Hom}_k(M,N)$ be the natural R-R bimodule. Then there exists a natural isomorphism of k-vector spaces*

$$\sigma_n : \mathrm{Ext}^n_R(M,N) \to \mathrm{HH}^n(R,\mathrm{Hom}_k(M,N))$$

for all $n \geq 0$.

Proof. From Weibel [**33**], lemma 9.1.9, we have a natural isomorphism of k-vector spaces between the Hochschild cohomology group $\mathrm{HH}^n(R, \mathrm{Hom}_k(M, N))$ and the relative Ext group $\mathrm{Ext}^n_{R/k}(M, N)$ for $n \geq 0$. But since k is a field, there is a canonical isomorphism between the relative and absolute Ext groups, see theorem 8.7.10 in Weibel [**33**]. $\square$

We shall give an explicit identification of k-vector spaces between $\mathrm{Ext}^1_R(M, N)$ and $\mathrm{HH}^1(R, \mathrm{Hom}_k(M, N))$: Let (L_*, d_*) be a free resolution of M, with augmentation morphism $\rho : L_0 \to M$, and let $\tau : M \to L_0$ be a k-linear section of ρ. For any 1-cocycle $\phi \in \mathrm{Hom}_R(L_1, N)$, let $\psi = \psi(\phi) \in \mathrm{Der}_k(R, \mathrm{Hom}_k(M, N))$ be the following derivation: For any $r \in R,\ m \in M$, let $x = x(r, m) \in L_1$ be such that $d_0(x) = r\tau(m) - \tau(rm)$. Notice that such an x exists, and is uniquely defined modulo the image $\mathrm{Im}\, d_1$. We define ψ by the equation $\psi(r)(m) = \phi(x)$ with $x = x(r, m)$. Since ϕ is a cocycle, ψ is a well-defined homomorphism in $\mathrm{Hom}_k(R, \mathrm{Hom}_k(M, N))$, and a straight-forward calculation shows that ϕ is a derivation.

Lemma A.4. *Assume that* $\mathrm{Ext}^1_R(M, N)$ *is a finite dimensional k-vector space. Then the assignment* $\phi \mapsto \psi(\phi)$ *defined in the above paragraph induces the isomorphism* $\sigma_1 : \mathrm{Ext}^1_R(M, N) \to \mathrm{HH}^1(R, \mathrm{Hom}_k(M, N))$.

Proof. Assume that ϕ is a co-boundary, so $\phi = d^0(\phi')$, where $\phi' \in \mathrm{Hom}_R(L_0, N)$. Then $\psi = d^0(\phi')$, where $\psi' = \phi'\tau \in \mathrm{Hom}_k(M, N)$, so ϕ is a trivial derivation. Consequently, the assignment induces a well-defined map of k-linear spaces. This map is furthermore injective: Assume that ψ is a trivial derivation, so $\psi = d^0(\psi')$, where $\psi' \in \mathrm{Hom}_k(M, N)$. Then, we can construct an R-linear map $\phi' \in \mathrm{Hom}_R(L_0, N)$ in the following way: Choose a basis for L_0, and for each basis vector $y \in L_0$, choose $y' \in L_1$ such that $d_0(y') = y - \psi'\rho(y)$. Then we define $\phi'(y) = \psi'\rho(y) + \phi(y')$ for each basis vector $y \in L_0$. We obtain a morphism $\phi' \in \mathrm{Hom}_R(L_0, N)$ by R-linear extension, and $d^0(\phi') = \phi$, so ϕ is a co-boundary. But since $\dim_k \mathrm{Ext}^1_R(M, N)$ is finite, it equals $\dim_k \mathrm{HH}^1(R, \mathrm{Hom}_k(M, N))$ by proposition A.3. This shows that the k-linear map defined above is an isomorphism, and it coincides with σ_1 by naturality. $\square$

4. Extensions and Hochschild cohomology

Let R be an associative k-algebra, and let M, N be left R-modules. An extension of M with N is an exact sequence

$$0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0.$$

Furthermore, we say that two extensions E, E' of M with N are equivalent if there is an isomorphism of left R-modules $\phi : E \to E'$ such that $\phi f = f'$ and $g'\phi = g$. The set of equivalence classes of extensions of M with N is denoted $\mathrm{Ext}(M, N)$.

Proposition A.5. *Let M, N be left R-modules. Then, there is a bijective correspondence between the set* $\mathrm{Ext}(M, N)$ *of extensions of M with N and the Hochschild cohomology* $\mathrm{HH}^1(R, \mathrm{Hom}_k(M, N))$. *In particular, the set* $\mathrm{Ext}(M, N)$ *is isomorphic to* $\mathrm{Ext}^1_R(M, N)$, *and carries a natural structure of a k-vector space.*

Proof. Let $0 \to N \to E \to M \to 0$ be an extension in $\mathrm{Ext}(M, N)$. Then E has an underlying structure as a k-vector space, and $E \cong N \oplus M$ as k-vector spaces. We may therefore assume that $E = N \oplus M$ over k, with some left multiplication by R given, and that the maps $f : N \to E$ and $g : E \to M$ are the natural ones. Since f, g are R-linear maps, we have that $r(n, m) = (rn + \psi(r, m), rm)$ for all $r \in R,\ n \in N,\ m \in M$ for some map $\psi : R \times M \to N$. Distributivity gives that

ψ is k-linear in R and M, and associativity gives that ψ is a derivation. So there is a bijection between extensions E and derivations $\psi \in \mathrm{Der}_k(R, \mathrm{Hom}_k(M,N))$. We claim that two extensions E, E' are equivalent if and only if the corresponding derivations ψ, ψ' represent the same class in $\mathrm{HH}^1(R, \mathrm{Hom}_k(M,N))$: Assume that $\phi : E \to E'$ is an equivalence of extensions. Then $\phi : N \oplus M \to N \oplus M$, and ϕ has the form $\phi(n,m) = (n + \gamma(m), m)$ for all $n \in N,\ m \in M$ for some map $\gamma : M \to N$. This follows, since ϕ is an equivalence of extensions. Clearly, γ is k-linear since ϕ is, and the R-linearity of ϕ is equivalent to the equation

$$r\gamma(m) + \psi'_r(m) = \gamma(rm) + \psi_r(m)$$

for all $r \in R,\ m \in M$. So ϕ is R-linear if and only if $d^0(\gamma) = \psi - \psi'$. The inverse of ϕ is easily obtained, it is given by $(n,m) \mapsto (n - \gamma(m), m)$, so this proves that E and E' are equivalent extensions if and only if the corresponding derivations differ by a trivial derivation. □

5. Cup products on Ext groups

Let R be an associative k-algebra, and let M, N, P be left R-modules. There is a uniquely defined cup product on Ext groups

$$c : \mathrm{Ext}^1_R(N,P) \otimes_k \mathrm{Ext}^1_R(M,N) \to \mathrm{Ext}^2_R(M,P),$$

which is a k-linear map. We shall write $\xi \cup \eta$ for the cup product $c(\xi \otimes \eta)$, when $\xi \in \mathrm{Ext}^1_R(N,P),\ \eta \in \mathrm{Ext}^1_R(M,N)$. We shall recall two different ways of computing this cup product, using the Yoneda complex and the Hochschild complex:

Let us first assume that $\phi(\xi) \in \mathrm{Hom}^1(L'_*, L''_*)$ is a representative of ξ and that $\phi(\eta) \in \mathrm{Hom}^1(L_*, L'_*)$ is a representative of η in the corresponding Yoneda complexes, with L_*, L'_*, L''_* free resolutions of M, N, P. Then $\phi(\xi)_i : L'_{i+1} \to L''_i$ and $\phi(\eta)_i : L_{i+1} \to L'_i$ for $i \geq 0$. Let $\phi' \in \mathrm{Hom}^2(L_*, L''_*)$ be given by the formula

$$\phi'_i = \phi(\xi)_i \circ \phi(\eta)_{i+1}$$

for $i \geq 0$. A straight-forward calculation shows that ϕ' is a 2-cocycle, since $\phi(\xi), \phi(\eta)$ are 1-cocycles. Moreover, the cohomology class of ϕ' in $H^2(\mathrm{Hom}(L_*, L''_*))$ is independent upon the choice of representative $\phi(\xi)$ in $H^1(\mathrm{Hom}(L'_*, L''_*))$ and of representative $\phi(\eta)$ in $H^1(\mathrm{Hom}(L_*, L'_*))$. So the map $\phi(\xi) \otimes \phi(\eta) \mapsto \phi'$ induces a k-linear map on cohomology, and this is the cup product.

Assume that M, N, P have free resolutions L_*, L'_*, L''_* such that L_i, L'_i, L''_i are finite free R-modules for all $i \geq 0$. Then we may choose a finite basis for each of these finite free R-modules, and identify each $\phi(\xi)_i$ with a matrix A_i, and each $\phi(\eta)_i$ with a matrix B_i. We recall that this identification is given by matrix multiplication from the right. Using this identification, we see that each ϕ'_i is identified with a matrix $C_i = B_{i+1}A_i$.

Secondly, let us assume that $\psi(\xi) \in \mathrm{Der}_k(R, \mathrm{Hom}_k(N,P)$ is a representative of ξ, and that $\psi(\eta) \in \mathrm{Der}_k(R, \mathrm{Hom}_k(M,N))$ is a representative of η in the Hochschild complex. We define $\psi' \in \mathrm{Hom}_k(R, \mathrm{Hom}_k(M,P))$ to be given by the formula

$$\psi'(r \otimes s) = \psi(\xi)(r) \circ \psi(\eta)(s)$$

for all $r, s \in R$. A straight-forward calculation shows that ψ' is a derivation, since $\psi(\xi), \psi(\eta)$ are derivations. Furthermore, we see that the cohomology class of ψ' in $\mathrm{HH}^2(R, \mathrm{Hom}_k(M,P))$ is independent upon the choice of representative $\psi(\xi)$ in $\mathrm{HH}^1(R, \mathrm{Hom}_k(N,P))$ and of representative $\psi(\eta)$ in $\mathrm{HH}^1(R, \mathrm{Hom}_k(M,N))$. So the map $\psi(\xi) \otimes \psi(\eta) \mapsto \psi'$ induces a k-linear map on cohomology, and this is the cup product.

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