D-MODULES ON SMOOTH ALGEBRAIC VARIETIES

EIVIND ERIKSEN

Abstract. We consider algebraic varieties X defined over $\mathbf{C}$ which are smooth, affine and irreducible. We study the ring $D = D(X)$ of $\mathbf{C}$-linear differential operators on X, and we explain Bernstein's theory of holonomic D-modules in this case. This is a generalization of Bernstein's original work, which covers the case when X is affine n-space and D is the n'th Weyl algebra. I shall follow the approach to this generalized theory given in chapter 3 of Björk [1].

1. Differential operators on the variety X

Let $k = \mathbf{C}$ denote the field of complex numbers. We consider an affine algebraic variety $X \subseteq \mathbf{C}^n$ such that X is smooth and irreducible. We denote by $A = A(X)$ the affine coordinate ring of X, so A is a commutative k-algebra of finite type and an integral domain. In particular, A is a Notherian ring.

Let $k(X)$ denote the quotient field of the integral domain A. Since A is of finite type over k, it is clear that $k \subseteq k(X)$ is a field extension of finite degree of transcendence. We denote by $d = \dim(X)$ the degree of transcendence of this field extension. This is the classically defined dimension of the variety X. It is clear that A is a regular Noetherian ring of pure dimension d. That is, the local Noetherian ring A_m is a regular ring of dimension d for all maximal ideals $m \subseteq A$. So clearly, the Krull dimension $\dim A = d$, and we have $0 \leq d \leq n$ with

- $d = n$ if and only if $X = \mathbf{C}^n$,
- $d = 0$ if and only if X is reduced to a single point.

The vector fields on X are given as $\theta(X) = \mathrm{Der}_k(A)$, where $\mathrm{Der}_k(A)$ is the module of derivations

$$\mathrm{Der}_k(A) = \{D \in \mathrm{End}_k(A) : D(xy) = D(x)y + xD(y) \text{ for all } x, y \in A\}.$$

Let $m \subseteq A$ be any maximal ideal, let A_m be the corresponding local ring and let $t_1, \dots, t_d$ be a local system of parameters for A_m. Then $\mathrm{Der}_k(A_m) \cong A_m \otimes_A \mathrm{Der}_k(A)$ is a free A_m-module of rank d, generated by derivations $D_1, \dots, D_d$ such that $D_i(t_j) = \delta_{ij}$ for $1 \leq i, j \leq d$.

We may present the ring A in the form $A = S/I$, where $S = k[x_1, \dots, x_n]$ is the affine coordinate ring of $\mathbf{C}^n$ and $I = I(X) \subseteq S$ is the prime ideal in S consisting of all polynomials in S which vanish on X. It is clear that $\mathrm{Der}_k(S)$ is the free S-module generated by $\partial_i = \partial/\partial x_i$ for $1 \leq i \leq n$. It is not difficult to see that there is a canonical isomorphism

$$\mathrm{Der}_k(A) \cong \{P \in \mathrm{Der}_k(S) : P(I) \subseteq I\}/I\,\mathrm{Der}_k(S),$$

and that $P(I) \subseteq I$ is satisfied if and only if $P(f_i) \in I$ for any set of generators $f_1, \dots, f_r$ of the ideal I. So $\mathrm{Der}_k(A)$ can be identified with the kernel of the A-linear map $A^n \to A^r$ given by the matrix $(\partial f_i/\partial x_j)$. Since A is a Noetherian ring, it follows that $\mathrm{Der}_k(A)$ is a left A-module of finite type.

We define the ring $D = D(X)$ of k-linear differential operators on X to be the sub-ring of $\mathrm{End}_k(A)$ generated by the multiplication operators induced by the ring A and the derivations in $\mathrm{Der}_k(A)$. It follows that $D(X)$ is a associative k-algebra.

Since A is a finitely generated k-algebra and $\mathrm{Der}_k(A)$ is a finitely generated A-module, it follows that D is a k-algebra of finite type.

We denote by $D^p \subseteq D$ the k-linear subspace of D generated by products of at most p derivations for any integer p. Then $D^p = 0$ when $p < 0$, $D^0 = A$, and $D^1 = A \oplus \mathrm{Der}_k(A)$. Moreover, we have that the subspaces D^p form an exhaustive, ascending filtration of the ring D. This filtration is called the order filtration, and we say that a differential operator $P \in D$ has order p if $P \in D^p \setminus D^{p-1}$ for some $p \geq 0$, and that $P = 0$ has order $-\infty$. We shall write $d(P)$ for the order of the differential operator P. Note that the filtered ring D coincides with the ring of differential operators on X/k defined by Grothendieck in EGA IV [2].

Consider the associated graded ring $\mathrm{gr}\, D$ associated with the order filtration of the ring D, defined as

$$\mathrm{gr}\, D = \oplus D^p/D^{p-1}.$$

This is a graded k-algebra. We shall denote by $\mathrm{gr}^p\, D$ the p'th homogeneous part D^p/D^{p-1} of $\mathrm{gr}\, D$ for all integers p. Then we have $\mathrm{gr}^p\, D = 0$ when $p < 0$, $\mathrm{gr}^0\, D = A$ and $\mathrm{gr}^1\, D = \mathrm{Der}_k(A)$. Since we have $d(PQ - QP) < d(P) + d(Q)$ for all non-zero differential operators P, Q, we see that $\mathrm{gr}\, D$ is a commutative ring. Moreover, it is a finitely generated k-algebra, and hence Noetherian, since it is generated by homogeneous elements of degree 1 considered as a $\mathrm{gr}^0\, D$-algebra.

In the following theorem, we summarize some properties of the rings D and $\mathrm{gr}\, D$ which will be useful. The proof of most of the statements in this theorem can be found in Björk [1], and references to the remaining parts can be found in Smith and Stafford [3]:

Theorem 1. *Let X be a smooth, irreducible affine algebraic variety of dimension d defined over $\mathbf{C}$, let D be the ring of differential operators and let $\mathrm{gr}\, D$ be the associated graded ring associated with the order filtration on D. Then we have:*

i) D is an associative k-algebra of finite type,
ii) D is an integral domain,
iii) D is a simple ring,
iv) D has global homological dimension d,
v) $\mathrm{gr}\, D$ is a commutative k-algebra of finite type,
vi) $\mathrm{gr}\, D$ is an integral domain,
vii) $\mathrm{gr}\, D$ is a Noetherian regular ring of pure dimension $2d$.

2. Modules on filtered rings

Let D be any filtered k-algebra with a fixed ascending filtration $\{D^p\}$ of k-linear subspaces of D. We shall assume that the filtration (D^p) is exhaustive and such that D^0 contains the unit $1 \in D$ and such that $D^p = 0$ for all $p < 0$. Moreover, we assume that D^p is finitely generated considered as a left and right D^0-module for all integers p. Finally, let us consider the associated graded ring $\mathrm{gr}\, D$, and assume that $\mathrm{gr}\, D$ is a commutative Noetherian ring. This last condition implies that D^0 is a commutative, Noetherian k-algebra.

Notice that when X is a smooth, irreducible affine algebraic variety over $\mathbf{C}$ and $D = D(X)$ is the ring of k-linear differential operators with the order filtration, then these conditions are fulfilled. Moreover, the ring $D^0 = A$, the affine coordinate ring of X. This example will motivate the constructions in this section.

We refer to any element $P \in D$ as an operator, and we denote by $d(P)$ the order of the operator P, defined as $d(P) = \inf\,\{p : P \in D^p\}$. By convention, $d(P) = -\infty$ when $P = 0$. When $P \in D$, we denote by $\sigma(P)$ the image of P in $D^p/D^{p-1} \subseteq \mathrm{gr}\, D$ with $p = d(P)$. By convention, we have $\sigma(P) = 0$ when $P = 0$.

Let M be a left D-module. We denote by a *filtration* of M any exhaustive, ascending filtration $\{M_i\}$ of M compatible with the given filtration of D such that

M_i is a finitely generated D^0-module for all integers i and $M_i = 0$ for some integer i. For any such filtration, we consider the associated graded $\operatorname{gr} D$-module

$$\operatorname{gr} M = \oplus M_i/M_{i-1}.$$

For any element $m \in M$, we denote by $d(m) = \inf\{i : m \in M_i\}$ the order of the element m. By convention, $d(m) = -\infty$ when $m = 0$. We denote by $\sigma(m)$ the image of m in $M_i/M_{i-1} \subseteq \operatorname{gr} M$ with $i = d(m)$. By convention, $\sigma(m) = 0$ when $m = 0$.

Proposition 2. *Let M be a left D-module, and let (M_i) be a filtration of M. If $\{m_\alpha\} \subseteq M$ is a subset of M such that $\{\sigma(m_\alpha)\}$ is a generating set for $\operatorname{gr} M$ as a left $\operatorname{gr} D$-module, then $\{m_\alpha\}$ is a generating set for M as a left D-module.*

Proof. Assume that $\{\sigma(m_\alpha)\}$ is a generating set of $\operatorname{gr} M$, and let $\overline{M} \subseteq M$ denote the left D-module generated by $\{m_\alpha\}$. It is enough to show that $M_i \subseteq \overline{M}$ for all integers i. Since $M_i = 0$ for some integer i, we can prove this by induction on i. So assume that $M_{i-1} \subseteq \overline{M}$, and let $m \in M_i \setminus M_{i-1}$. Then we have

$$\sigma(m) = \sum \sigma(P_\alpha)\sigma(m_\alpha)$$

for operators P_α of degree $i - d(m_\alpha)$. If follows that $m - \sum P_\alpha m_\alpha \in M_{i-1}$. By the induction hypothesis, $M_{i-1} \subseteq \overline{M}$, so it follows that $M_i \subseteq \overline{M}$. □

Assume that M is a finitely generated left D-module, and choose a finite set $\{m_\alpha\}$ of generators for M. We define $M_i = \sum D^i m_\alpha$ for all integers i. Then (M_i) is a filtration of M, and $\{\sigma(m_\alpha)\}$ is a finite generating set for $\operatorname{gr} M$ considered as a $\operatorname{gr} D$-module. This proves the following proposition:

Proposition 3. *Let M be a left D-module. Then there exists a filtration of M such that $\operatorname{gr} M$ is a finitely generated $\operatorname{gr} D$-module if and only if M is a finitely generated D-module.*

Corollary 4. *The ring D is left Noetherian.*

Proof. Let $I \subseteq D$ be a left ideal. Then $I \subseteq D$ is a left sub-module. Consider the filtration (I^p) with $I^p = I \cap D^p$ for all integers p. Then the inclusion $I \subseteq D$ induces an inclusion $\operatorname{gr} I \subseteq \operatorname{gr} D$, and in particular, $\operatorname{gr} I \subseteq \operatorname{gr} D$ is an ideal. Since $\operatorname{gr} D$ is a Noetherian ring, this is a finitely generated ideal. By the above proposition, this means that I is a finitely generated left D-module, hence a finitely generated ideal in D. It follows that D is a left Noetherian ring. □

Let M be a left D-module, and let (M_i) be a filtration of M. We say that (M_i) is a *good filtration* if the associated graded $\operatorname{gr} D$-module $\operatorname{gr} M$ is finitely generated. By the above proposition, there exists a good filtration of any finitely generated left D-module. We show the following strong result on their uniqueness:

Proposition 5. *Let M be a left D-module, and let $(M_i), (M_i')$ be good filtrations of M. Then there exists a non-negative integer w such that $M'_{i-w} \subseteq M_i \subseteq M'_{i+w}$ for all integers i.*

Proof. It is enough to show that there exists a non-negative integer w such that $M_i \subseteq M'_{i+w}$ for all integers i. We may also assume that M_i is a filtration such that $M_i = 0$ when $i < 0$. Denote by $\operatorname{gr} M$ the graded $\operatorname{gr} D$-module associated with the filtration (M_i). This is by definition a finitely generated $\operatorname{gr} D$-module, so we may find an integer $v \geq 0$ such that

$$\operatorname{gr} M_{\leq v} = \bigoplus_{i \leq v} M_i/M_{i-1}$$

generates $\operatorname{gr} M$. We define the k-vector space $N_i = D^i M_0 + \cdots + D^{i-v} M_v \subseteq M_i$ for all $i \geq v$. Clearly, we have $N_v = M_v$ since $1 \in D^0$. Let $m \in M_i \setminus M_{i-1}$ for some $i > v$. Then we have

$$\sigma(m) \in (D^i/D^{i-1})(M_0/M_{-1}) + \cdots + (D^{i-v}/D^{i-v-1})(M_v/M_{v-1}),$$

so $M_i \subseteq N_i + M_{i-1}$ for all $i > v$. By induction, this gives $M_i = N_i$ for all $i \geq v$. Consider the filtration of M_v given by k-linear subspaces $M_v \cap M'_i$. This filtration is exhaustive and each M'_i is finitely generated as left D^0-module. Hence there exists an integer w such that $M_v \subseteq M'_w$. We see that if $i < v$, we have

$$M_i \subseteq M_v \subseteq M'_w \subseteq M'_{i+w}.$$

If $i \geq v$, then for each integer j with $0 \leq j \leq v$, we have that

$$D^{i-j} M_j \subseteq D^{i-j} M_v \subseteq D^{i-j} M'_w \subseteq D^i M'_w \subseteq M'_{i+w}.$$

This means that $M_i = N_i \subseteq M'_{i+w}$ for all integers $i \geq v$. So we have proved that $M_i \subseteq M'_{i+w}$ for all integers i. □

Let M be a left D-module, and (M_i) a chosen good filtration of M. We consider the associated graded ring $\operatorname{gr} M$ with respect to this chosen filtration. This is a finitely generated $\operatorname{gr} D$-module by definition. Recall that $\operatorname{gr} D$ is a commutative Noetherian k-algebra, and let $m \subseteq \operatorname{gr} D$ be a maximal ideal. Then $(\operatorname{gr} D)_m$ is a local Noetherian ring, and $(\operatorname{gr} M)_m = (\operatorname{gr} D)_m \otimes_{\operatorname{gr} D} \operatorname{gr} M$ is a finitely generated $(\operatorname{gr} D)_m$-module. Then there exists a uniquely defined Hilbert-Zariski-Samuel polynomial for the $(\operatorname{gr} D)_m$-module $(\operatorname{gr} M)_m$, and we may define the local dimension $d_m(\operatorname{gr} M)$ and the local multiplicity $e_m(\operatorname{gr} M)$ of $\operatorname{gr} M$ at the maximal ideal m. We shall show that these local invariants do not depend on the chosen good filtration of M:

Proposition 6. *Let M be a left D-module, let $(M_i), (M'_i)$ be good filtrations of M, and let $\operatorname{gr} M, \operatorname{gr}' M$ be the corresponding $\operatorname{gr} D$-modules. Then for all maximal ideals $m \subseteq \operatorname{gr} D$, we have $d_m(\operatorname{gr} M) = d_m(\operatorname{gr}' M)$ and $e_m(\operatorname{gr} M) = e_m(\operatorname{gr}' M)$.*

Proof. From the previous proposition, we have that $M'_{i-w} \subseteq M_i \subseteq M'_{i+w}$ for some integer w. Clearly, the local invariants are not changed by shifts, so we may assume that $M'_i \subseteq M_i \subseteq M'_{i+w}$ for all integers i by a shift in the filtration M_i, if necessary. We shall define a sequence of good filtrations of M, (T^p_i) for $0 \leq p \leq w$, with the following properties: $T^0_i = M'_i$, $T^w_i = M_i$ for all integers i, $T^p_i \subseteq M_i$ for all integers i, p, and the good filtrations (T^p_i) give the same local invariants for $0 \leq p \leq w$. This construction would clearly prove the proposition. We put $T^0_i = M'_i$ for all integers i, and we define the filtrations (T^p_i) by induction on p. So assume that good filtrations $(T^0_i), \ldots, (T^{p-1}_i)$ are defined with the required properties. We define $T^p_i = M_i \cap T^{p-1}_{i+1}$ for all integers i. Since (T^{p-1}_i) is a filtration and D^0 is a commutative Noetherian ring, it is clear that (T^p_i) is a filtration as well. We have to show that (T^p_i) is a good filtration.

We see that $T^{p-1}_i \subseteq T^p_i \subseteq T^{p-1}_{i+1}$ for all integers i. So there are short exact sequences

$$0 \to T^p_i/T^{p-1}_i \to T^{p-1}_{i+1}/T^{p-1}_i \to T^{p-1}_{i+1}/T^p_i \to 0$$

and

$$0 \to T^{p-1}_{i+1}/T^p_i \to T^p_{i+1}/T^p_i \to T^p_{i+1}/T^{p-1}_{i+1} \to 0$$

of k-vector spaces. Let $Z^p = \oplus T^p_i/T^{p-1}_i$ and $B^p = T^{p-1}_{i+1}/T^p_i$, then Z^p and B^p are graded $\operatorname{gr} D$-modules. We denote by $\operatorname{gr}^{(p)} M$ the graded $\operatorname{gr} D$-module associated with the filtration (T^p_i) for all integers p. Then we have exact sequences

$$0 \to Z^p \to \operatorname{gr}^{(p-1)} M \to B^p \to 0$$

and

$$0 \to B^p \to \operatorname{gr}^{(p)} M[1] \to Z^p[1] \to 0$$

of graded $\operatorname{gr} D$-modules. Since (T_i^{p-1}) is a good filtration and $\operatorname{gr} D$ is Noetherian, all modules in the first exact sequence are finitely generated $\operatorname{gr} D$-modules. Since the property of being finitely generated, graded $\operatorname{gr} D$-modules is independent upon shifts, it follows that all modules in the second exact sequence are finitely generated $\operatorname{gr} D$-modules as well. Consequently, we see that (T_i^p) is a good filtration as well. We also see from the exact sequences given above that the good filtrations (T_i^{p-1}) and (T_i^p) give the same local invariants, since these invariants are independent upon shifts.

It only remains to see that $T_i^w = M_i$ for all integers i. But an easy induction argument shows that $T_i^p = M_i \cap T_{i+j}^{p-j}$ for all integers j with $0 \leq j \leq p$. With $p = w$, this gives $T_i^w = M_i \cap T_{i+w}^0 = M_i \cap M'_{i+w} = M_i$ for all integers i. □

Let M be a left D-module of finite type, and let $m \subseteq \operatorname{gr} D$ be a maximal ideal. We define the local dimension $d_m(M)$ and the local multiplicity $e_m(M)$ of M to be the local dimension and multiplicity of the associated graded module $\operatorname{gr} M$ with respect to some good filtration of the D-module M. By the above proposition, these invariants are independent upon the choice of good filtration of M.

Let M be a left D-module of finite type. We define the dimension of M to be $d(M) = \sup\{d_m(M) : m \subseteq \operatorname{gr} D \text{ is a maximal ideal}\}$, and the multiplicity of M to be $e(M) = \inf\{e_m(M) : m \subseteq \operatorname{gr} D \text{ is a maximal ideal such that } d_m(M) = d(M)\}$. We easily deduct the following properties of these invariants:

Proposition 7. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated left D-modules. Then $d(M) = \sup\{d(M'), d(M'')\}$. Moreover, if $d(M') = d(M'')$, then $e(M) = e(M') + e(M'')$.*

Let M be a left D-module, and let (M_i) be a good filtration of M. We consider the associated graded $\operatorname{gr} D$-module $\operatorname{gr} M$ associated with (M_i). Let $J(M)$ be the radical of the annihilator ideal $a = \operatorname{ann}_{\operatorname{gr} D} \operatorname{gr} M \subseteq \operatorname{gr} D$. Then $J(M)$ is a radical, graded ideal in $\operatorname{gr} D$, and we show that it does not depend upon the chosen good filtration of M:

Proposition 8. *Let M be a left D-module, let $(M_i), (M'_i)$ be good filtrations of M, and let $\operatorname{gr} M, \operatorname{gr}' M$ be the associated $\operatorname{gr} D$-modules. We denote by $J(M), J'(M)$ the radicals of the corresponding annihilator ideals. Then $J(M) = J'(M)$.*

Proof. It is clearly enough to prove that $J(M) \subseteq J'(M)$, and we may show this inclusion by considering homogeneous elements. So let $\sigma(P) \in J(M)$ be an homogeneous element of degree d, and assume that $\sigma(P)^m \operatorname{gr} M = 0$. Then $P \in D^p \setminus D^{p-1}$, such that $P^m M_i \subseteq M_{i+md-1}$ for all integers. By iterating this equation q times, we get $P^{qm} M_i \subseteq M_{i+qmd-q}$ for all integers i. But since $(M_i), (M'_i)$ are good filtrations of M, we have $M_{i-w} \subseteq M'_i \subseteq M_{i+w}$ for all integers i. With $q = 2w + 1$, these equations give

$$P^{m(2w+1)} M'_i \subseteq P^{m(2w+1)} M_{i+w} \subseteq M_{i+md(2w+1)-w-1} \subseteq M'_{i+md(2w+1)-1}.$$

This means that $\sigma(P)^{m(2w+1)} \operatorname{gr}' M = 0$, so $\sigma(P) \in J'(M)$. □

We define the *characteristic variety* $\operatorname{Char}(M)$ of M to be the variety corresponding to the the radical ideal $J(M)$. This is an affine variety, with affine coordinate ring $\operatorname{gr} D/J(M)$. The closed points in this variety corresponds to the maximal ideals $m \subseteq \operatorname{gr} D$ such that $J(M) \subseteq m$, or equivalently such that $(\operatorname{gr} M)_m \neq 0$. We denote by $d(M)$ the Krull dimension of $\operatorname{gr} D/J(M)$, which equals the dimension of

$\text{Char}(M)$. Clearly, $d(M)$ also coincides with the dimension of M defined above via Hilbert-Zariski-Samuel polynomials.

We remark that all the result given in this section for left D-modules, hold equally well for right D-modules. This follows from the symmetry of the assumptions on the filtered ring D. We also note that all results hold equally well for an algebraically closed field k of characteristic 0. Moreover, if k is a field of characteristic 0 but not necessarily algebraically closed, all results in this section except the results on characteristic variety still hold.

3. The Weyl algebra $A_n(k)$

Consider the case $X = \mathbf{C}^n$. In this case, X is a smooth variety of dimension $d = n$, and its affine coordinate ring is the polynomial ring $A = k[x_1, \ldots, x_n]$. Clearly, the module of derivations on A is the free A-module with generators $\partial_1, \ldots, \partial_n$, where $\partial_i = \partial/\partial x_i$ for $1 \leq i \leq n$. It follows that the corresponding ring of differential operators $D = D(X)$ is the n'th Weyl-algebra $A_n(k)$, which has generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ and relations $[\partial_i, x_i] = 1$ for $1 \leq i \leq n$.

We may consider D a filtered ring, with the order filtration (D^p), as explained in the section 1. When we do, the results from theorem 1 applies. In particular, $\text{gr}\, D$ is a polynomial ring in $2n$ variables over k. It is isomorphic to $k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$, where ξ_i denotes the image of ∂_i in $\text{gr}\, D$ for $1 \leq i \leq n$.

We shall define another filtration on the ring D with similar properties, the Bernstein filtration (B^p): For any integer p, we define B^p to be the k-linear subspace of D generated by all differential operators of the form

$$x_1^{l_1} \ldots x_n^{l_n} \partial_1^{m_1} \ldots \partial_n^{m_n}$$

for integers $l_1, \ldots, m_n$ such that $l_1 + \cdots + m_n \leq p$. We immediately see that $B^p = 0$ when $p < 0$ and that $B^0 = k$. Furthermore, it is not hard to check that (B^p) is an ascending, exhaustive filtration of D such that B^p is a finite dimensional vector space over $B^0 = k$ for all integers p. A straight-forward computation shows that the associated graded ring with respect to the Bernstein filtration is a polynomial ring in $2n$ variables, isomorphic to $k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ as k-algebras.

We conclude that the ring $D = A_n(k)$ with the Bernstein filtration fulfills all the condition of section 2, so all results from this section applies to this ring. We also notice that the graded ring $\text{gr}\, D$ associated with the Bernstein filtration clearly is a regular Noetherian ring of pure dimension $2d = 2n$.

Let M be a finitely generated left D-module, and let (M_n) be a good filtration of M with respect to the Bernstein filtration of D. We consider the associated graded $\text{gr}\, D$-module $\text{gr}\, M$, and we denote by $d(M)$ its Krull dimension. By standard results about Hilbert functions, there exists a Hilbert polynomial $P_M \in \mathbf{Q}[t]$, which depends upon the chosen good filtration, such that $\dim_k M_i = P_M(i)$ for all $i >> 0$. Furthermore, this polynomial has leading term $e/d!t^d$, where $d = d(M)$ and e is a strictly positive integer which does not depend upon the chosen good filtration.

Consider any good filtration of a left D-module M compatible with the Bernstein filtration. We notice that the dimension of M defined via Hilbert polynomials and the dimension of M defined via Hilbert-Zariski-Samuel polynomials both equal the Krull dimension $d(M)$ of $\text{gr}\, M$, and hence these two dimensions coincide. We shall later see that the dimension $d(M)$ also coincides with the dimension of M defined via a good filtration compatible with the order filtration of D.

Lemma 9. *Let M be a left D-module of finite type, and let (M_i) be a good filtration of M such that $M_0 \neq 0$. Then the map $B^p \to \text{Hom}_k(M_p, M_{2p})$ is injective for all integers $p \geq 0$.*

Proof. For $p = 0$, the claim follows from $M_0 \neq 0$. Let us prove the claim by induction on p, so assume that the claim holds for $p-1$ when $p > 0$, and assume that $PM_p = 0$ for some $P \in B^p$. Then clearly $[P, x_i]M_{p-1} = [P, \partial_i]M_{p-1} = 0$, so P is in the centre of D by the induction hypothesis. But the centre of D is k and $M_p \neq 0$ since $M_0 \subseteq M_p$, so this means that $P = 0$. □

Theorem 10. *Let M be a non-zero left D-module of finite type. Then $d(M) \geq n$.*

Proof. Clearly, we can choose a good filtration of M such that $M_0 \neq 0$. Let P_M be the corresponding Hilbert polynomial. We know that the Hilbert polynomial corresponding to the Bernstein filtration of the D-module $M = D$ has leading coefficient $1/(2n)!\ t^{2n}$. So it follows from the previous lemma that the degree of P_M is at least n, since $\dim_k B^p \leq (\dim_k M_p)(\dim_k M_{2p})$ for all $p \geq 0$. □

We say that any finitely generated D-module M such that $M \neq 0$ and $d(M) = n$ or such that $M = 0$ is *holonomic.* From elementary facts about additivity of Hilbert functions along exact sequences, we see that any extension of holonomic modules is holonomic. From the previous theorem, it also follows that sub-modules and quotients of holonomic modules are holonomic.

Corollary 11. *Let M be a holonomic D-module. Then M is an Artinian and cyclic D-module.*

Proof. It is clear that any chain of submodules of M consists of holonomic modules. Since the multiplicity e is strictly smaller for a sub-module of a holonomic module, it follows that the multiplicity e of M is an upper bound for the length of M. This means that M is Artinian. But D is clearly not a left Artinian ring. We shall later see that this means that M is cyclic. □

Let λ be a fixed transcendental element over $A = k[x_1, \dots, x_n]$, and let $k(\lambda)$ be the field extension of $k = \mathbf{C}$ generated by λ. It is not difficult to see that the results of this section still hold over the field $k(\lambda)$. In particular, we may define the category of holonomic modules over the ring $D(\lambda) = A_n(k(\lambda))$.

Let $f \in A$ be a fixed polynomial of degree m. We consider the left $D(\lambda)$-module $M = k(\lambda)[x_1, \dots, x_n][1/f]f^\lambda$, where we consider f^λ a formal symbol acted on by the derivation ∂_i according to the formula

$$\partial_i f^\lambda = \lambda/f\ \partial_i(f)\ f^\lambda$$

for $1 \leq i \leq n$. We shall consider the functional equation

$$P(\lambda)f^{\lambda+1} = B(\lambda)f^\lambda,$$

where $P(\lambda) \in D[\lambda]$ and $B(\lambda) \in k[\lambda]$. It is clear that all polynomials $B(\lambda)$ which satisfy this functional equation for some $P(\lambda)$ form an ideal in $k[\lambda]$, and we denote this ideal by $b \subseteq k[\lambda]$. If this ideal is non-zero, there exists a unique monic polynomial $b(\lambda) \in k[\lambda]$ such that $b(\lambda)$ generates the ideal b. In this case, we say that $b(\lambda)$ is a *Bernstein polynomial* for $f \in A$.

Lemma 12. *Let M be a left $D(\lambda)$-module, and let (M_i) be a filtration of M such that $\dim_k M_i \leq c/n!\ i^n + c'(i+1)^{n-1}$ for some positive integers c, c'. Then M is holonomic. In particular, M is a finitely generated $D(\lambda)$-module.*

Theorem 13. *Let $f \in A = k[x_1, \dots, x_n]$ be a polynomial. Then there exists a Bernstein-polynomial $b(\lambda) \in k[\lambda]$ of f.*

Proof. Consider the $D(\lambda)$-module M defined above, and let M_i be the k-linear subspace of the form

$$M_i = \{q/f^i\ f^\lambda : \deg(q) - mi \leq i\}$$

for all integers i, where m is the degree of f. Then it is easy to see that M_i is a filtration of M, and we have that

$$\dim_k M_i = \binom{n-i(m+1)}{n},$$

so by the previous lemma, M is a holonomic $D(\lambda)$-module. It follows that the cyclic sub-module $N \subset M$ generated by f^λ is holonomic, and hence of finite length. Therefore the descending chain $N = N_0 \supseteq N_1 \supseteq \dots$, where N_l is the cyclic sub-module generated by $f^{\lambda+l}$, is stationary. It follows that there exists a differential operator $P \in D(\lambda)$ such that $f^{\lambda+l} = Pf^{\lambda+l+1}$. We may substitute $\lambda + l$ with λ, since λ is transcendent over k. Clearing the denominators of P shows that the ideal b is non-zero, and the result follows. □

4. Modules on filtered rings with regularity conditions

Let D be a filtered k-algebra. In this section, we shall assume that all the conditions of section 2 are fulfilled, and in addition that $\operatorname{gr} D$ is a regular Noetherian ring of pure dimension ω. This condition implies that $\operatorname{gr} D$ has global homological dimension ω and that D has a global homological dimension $\mu \leq \omega$.

Let M be a non-zero, left D-module of finite type. We may define the homological invariant $h(M)$ of M as

$$h(M) = \inf\,\{i \geq 0 : \operatorname{Ext}^i_D(M, D) \neq 0\}.$$

This invariant exists, and satisfy $0 \leq h(M) \leq \mu$. Furthermore, we denote by $d(M)$ the dimension of M defined via Hilbert-Zariski-Samuel polynomials. Then we have the following result:

Theorem 14. *Let M be a non-zero, left D-module of finite type, and assume that $\operatorname{gr} D$ is a regular Noetherian ring of pure dimension ω. Then $d(M) + h(M) = \omega$.*

Proof. See Björk [1], theorem 2.4.15, 2.5.7 and 2.7.1. □

We see that if D is an associative k-algebra with two distinct filtrations which satisfy the conditions of this section, then the dimension $d(M)$ defined via Hilbert-Zariski-Samuel polynomials is independent upon the chosen filtration of D. In particular, this applies to the ring $D(X) = A_n(k)$ corresponding to the variety $X = \mathbf{C}^n$: In this case, we may consider the order filtration or the Bernstein filtration of $D(X)$, and they both satisfy the conditions of the theorem. So this proves our claim from the previous section that the dimension $d(M)$ is independent upon which filtration we use.

Corollary 15. *Let M be a non-zero, left D-module of finite type, let μ be the global homological dimension of D, and assume that $\operatorname{gr} D$ is a regular Noetherian ring of pure dimension ω. Then $d(M) \geq \omega - \mu$.*

We say that a left D-module M of finite type is holonomic if $M = 0$ or if M is non-zero and $d(M) = \omega - \mu$. It follows from the results of the section 2 that extensions of holonomic modules are holonomic, and that sub-modules and quotients of holonomic modules are holonomic.

Theorem 16. *Let M be a finitely generated left D-module. If M is a holonomic D-module, then M is Artinian and cyclic.*

Proof. Clearly, any chain of sub-modules of M consists of holonomic modules, and the multiplicity $e(M)$ is strictly smaller for a holonomic sub-module. This means that the length of chains of sub-modules of M is bounded above by $e(M)$, and in particular M is Artinian. Since D is not left Artinian, we shall later see that M is cyclic. □

5. Holonomic D-modules

Let $X \subseteq \mathbf{C}^n$ be a smooth, irreducible affine algebraic variety of dimension d, and let $D = D(X)$ be the ring of differential operators on X, equipped with the order filtration. Then the filtered ring D satisfies the conditions of section 4. Moreover, the pure dimension of $\operatorname{gr} D$ is $\omega = 2d$, and the global homological dimension of D is $\mu = d$. So for every non-zero, left D-module M of finite type, we have $d(M) \geq d$ where $d = \dim X$. Furthermore, the category of holonomic D-modules consists of all D-modules M with $d(M) = d$ or $M = 0$. We have seen that any holonomic D-module M is Artinian and cyclic. The last implication uses the following theorem, which is due to Stafford (and which we used in section 3, as well):

Theorem 17. *Let R be any associative ring such that R is simple and such that R is not left Artinian. Then any Artinian left R-module is cyclic.*

References

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