Computing noncommutative deformations

Eivind Eriksen

Abstract Let M be a right module over an associative k-algebra A, where k is a field. We show how to compute noncommutative deformations of M in concrete terms, using an obstruction calculus based on free resolutions.

1 Introduction

Let A be an associative k-algebra, where k is a field. For any right A-module M, there is a noncommutative deformation functor $\mathsf{Def}_M : \mathsf{a}_1 \to \mathsf{Sets}$, introduced in Laudal [2], defined on the category a_1 of local Artinian k-algebras with residue field k. The noncommutative deformation functor extends the classical deformation functor $\mathsf{Def}^{\mathsf{cl}}_M : \mathbf{l} \to \mathsf{Sets}$, defined on the category $\mathbf{l}$ of local commutative Artinian k-algebras with residue field k.

In this paper, we show how to compute noncommutative deformations of M in concrete terms, using an obstruction calculus based on free resolutions. We show the computations explicitly in the example with $A = k[x,y]$ and $M = A/(x^2,y)$, which is obstructed. We also compare the result with the classical deformations of M.

2 Noncommutative deformations of modules

Let M be a right module over an associative k-algebra A, where k is a field. Then there is a classical deformation functor $\mathsf{Def}^{\mathsf{cl}}_M : \mathbf{l} \to \mathsf{Sets}$, where $\mathbf{l}$ is the category of commutative Artinian local k-algebras with residue field k. We fix a free resolution $(L_\bullet, d_\bullet)$ of M. For any algebra R in $\mathbf{l}$, a lifting of complexes from $(L_\bullet, d_\bullet)$ to R is

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Eivind Eriksen, BI Norwegian Business School
Department of Economics, N-0442 Oslo, Norway; e-mail: eivind.eriksen@bi.no

a complex $(L^R_\bullet, d^R_\bullet)$ of R-A bimodules, with $L^R_m = R \otimes_k L_m$, such that the following diagram commutes:

$$\begin{array}{ccccccc} L^R_0 & \xleftarrow{d^R_0} & L^R_1 & \xleftarrow{d^R_1} & L^R_2 & \xleftarrow{d^R_2} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ L_0 & \xleftarrow[d_0]{} & L_1 & \xleftarrow[d_1]{} & L_2 & \xleftarrow[d_2]{} & \dots \end{array}$$

It is well-known that $\mathsf{Def}^{\mathsf{cl}}_M(R)$ can be identified with the set of equivalence classes of liftings of $(L_\bullet, d_\bullet)$ to R, and that $\mathsf{Def}^{\mathsf{cl}}_M$ has tangent space $\mathsf{t}(\mathsf{Def}^{\mathsf{cl}}_M) \cong \mathrm{Ext}^1_A(M,M)$ and an obstruction theory with cohomology $\{\mathrm{Ext}^p_A(M,M)\}$. When $d = \dim_k \mathrm{Ext}^1_A(M,M)$ and $r = \dim_k \mathrm{Ext}^2_A(M,M)$ are finite, there is an obstruction morphism

$$o^{\mathsf{cl}} : k[[s_1, s_2, \dots, s_r]] \to k[[t_1, t_2, \dots, t_d]]$$

such that $H^{\mathsf{cl}} = k[[t_1, \dots, t_d]]/(f^{\mathsf{cl}}_1, \dots, f^{\mathsf{cl}}_r)$ is a pro-representing hull of $\mathsf{Def}^{\mathsf{cl}}_M$, with $f^{\mathsf{cl}}_i = o^{\mathsf{cl}}(s_i)$ for $1 \le i \le r$. Its versal family is given by a lifting of complexes of $(L_\bullet, d_\bullet)$ to H^{cl}.

There is an extension of the classical deformation functor $\mathsf{Def}^{\mathsf{cl}}_M$ of M to a noncommutative deformation functor $\mathsf{Def}_M : \mathsf{a}_1 \to \mathsf{Sets}$, where a_1 is the category of local Artinian k-algebras with residue field k. This extension is due to Laudal [2]; see also Eriksen [1] for details. We remark that $\mathsf{Def}_M(R)$ can be identified with the set of equivalence classes of liftings of $(L_\bullet, d_\bullet)$ to R. When $d = \dim_k \mathrm{Ext}^1_A(M,M)$ and $r = \dim_k \mathrm{Ext}^2_A(M,M)$ are finite, there is an obstruction morphism

$$o : k\langle\langle s_1, s_2, \dots, s_r \rangle\rangle \to k\langle\langle t_1, t_2, \dots, t_d \rangle\rangle$$

such that $H = k\langle\langle t_1, t_2, \dots, t_d \rangle\rangle/(f_1, \dots, f_r)$ is a pro-representing hull of Def_M, with $f_i = o(s_i)$ for $1 \le i \le r$. Its versal family is given by a lifting of complexes of $(L_\bullet, d_\bullet)$ to H.

The relationship between classical and noncommutative deformations are given by the following commutative diagram

$$\begin{array}{ccc} k\langle\langle s_1, s_2, \dots, s_r \rangle\rangle & \xrightarrow{o} & k\langle\langle t_1, t_2, \dots, t_d \rangle\rangle \\ \downarrow & & \downarrow \\ k[[s_1, s_2, \dots, s_r]] & \xrightarrow{o^{\mathsf{cl}}} & k[[t_1, t_2, \dots, t_d]] \end{array}$$

where the vertical maps are the natural commutativization homomorphisms given by $A \to A^{\mathsf{cl}} = A/(xy - yx : x, y \in A)$. In particular, f^{cl}_i is the image of f_i in $k[[t_1, \dots, t_d]]$ for $1 \le i \le r$.

3 A concrete description of lifting of complexes

Let R be an algebra in a_1, and choose a k-linear base $\{r_i : 0 \leq i \leq l\}$ of R with $r_0 = 1$. Then a lifting of complexes of $(L_\bullet, d_\bullet)$ to R is given by R-A linear maps $d^R_m : L^R_{m+1} \to L^R_m$ for $m \geq 0$, and d^R_m is determined by its value on elements of the form $1 \otimes f$ in $L^R_{m+1} = R \otimes_k L_{m+1}$. Therefore the differential d^R_m can be considered as an element in $\mathrm{Hom}_A(L_{m+1}, R \otimes_k L_m) \cong R \otimes_k \mathrm{Hom}_A(L_{m+1}, L_m)$, described in concrete terms as

$$d^R_m = 1 \otimes d_m + \sum_{i=1}^{l} r_i \otimes \alpha(r_i)_m$$

where $\underline{\alpha} = \{\alpha(r_i)_m : m \geq 0,\ 0 \leq i \leq l\}$ is a family of A-linear homomorphisms $\alpha(r_i)_m : L_{m+1} \to L_m$ with $\alpha(1)_m = d_m$. Conversely, such a family $\underline{\alpha}$ of A-linear homomorphisms represents a lifting of complexes of $(L_\bullet, d_\bullet)$ to R if and only if $d^R_m \circ d^R_{m+1} = 0$ for all $m \geq 0$. This condition can be expressed in terms of $\underline{\alpha}$ as

$$\sum_{1 \leq i \leq l} r_i \otimes (\alpha(r_i)_m \, d_{m+1} + d_m \, \alpha(r_i)_{m+1}) + \sum_{1 \leq i,j \leq l} r_j r_i \otimes \alpha(r_i)_m \, \alpha(r_j)_{m+1} = 0$$

Notice that on the tangent level, where $r_j r_i = 0$ for $1 \leq i, j \leq l$, $\underline{\alpha}$ determines a lifting of complexes to R if and only if $\alpha(r_i)$ is a 1-cocycle in the Yoneda complex $\mathsf{YC}^\bullet(L_\bullet, L_\bullet)$. We recall that the Yoneda complex $\mathsf{YC}^\bullet(L_\bullet, L_\bullet)$ is defined by

$$\mathsf{YC}^n(L_\bullet, L_\bullet) = \prod_{m \geq 0} \mathrm{Hom}_A(L_{m+n}, L_m)$$

for all $n \geq 0$, and with differential $d^n : \mathsf{YC}^n(L_\bullet, L_\bullet) \to \mathsf{YC}^{n+1}(L_\bullet, L_\bullet)$ given by

$$d^n(\phi)_m = \phi_m \, d_{n+m} + (-1)^{n+1} d_m \, \phi_{m+1} \quad \text{for } m \geq 0$$

for all $\phi = (\phi_m)_{m \geq 0} \in \mathsf{YC}^n(L_\bullet, L_\bullet)$. It is well-known that the cohomology of the Yoneda complex is $\mathsf{YH}^p(M, M) = \mathsf{H}^p(\mathsf{YC}^\bullet(L_\bullet, L_\bullet)) \cong \mathrm{Ext}^p_A(M, M)$.

4 Computing noncommutative deformations in an example

Let $A = k[x, y]$, and let M be the right A-module $M = A/(x^2, y)$ with free resolution $(L_\bullet, d_\bullet)$ given by

$$0 \leftarrow M \leftarrow A \xleftarrow{(x^2\ y)\cdot} A^2 \xleftarrow{\begin{pmatrix} y \\ -x^2 \end{pmatrix}\cdot} A \leftarrow 0$$

To compute $\mathrm{Ext}^p_A(M, M)$ for $p = 1$ (the tangent space) and $p = 2$ (the obstruction space), we consider the complex $\mathrm{Hom}_A(L_\bullet, M)$:

$$M \xrightarrow{\cdot(x^2\ y)} M^2 \xrightarrow{\cdot\begin{pmatrix} y \\ -x^2 \end{pmatrix}} M \to 0$$

Note that the differentials in this complex are zero. Since $M = k[x,y]/(x^2,y) \simeq k + kx$ has dimension two, we see that

$$\mathrm{Ext}^p_A(M,M) = \begin{cases} (k+kx)^2 \cong k^4, & p=1 \\ k+kx \cong k^2, & p=2 \end{cases}$$

Hence there are noncommutative power series $f_1, f_2 \in k\langle\langle t_1,t_2,t_3,t_4\rangle\rangle$ determined by the obstruction morphism such that $H = k\langle\langle t_1,t_2,t_3,t_4\rangle\rangle/(f_1,f_2)$ is a pro-representing hull of the noncommutative deformation functor Def_M. We shall compute f_1 and f_2 in concrete terms.

At the tangent level, $H_2 = k\langle\langle t_1,t_2,t_3,t_4\rangle\rangle/(t_1,t_2,t_3,t_4)^2$, and the versal family $\xi_2 \in \mathsf{Def}_M(H_2)$ is given by a lifting of complexes of $(L_\bullet, d_\bullet)$ to H_2. In concrete terms, the differential in $H_2 \otimes_k \mathrm{Hom}_A(L_{m+1},L_m)$ is given by

$$d_m^{H_2} = 1 \otimes d_m + \sum_{1\le i\le 4} t_i \otimes \alpha(t_i)_m$$

for all $m \geq 0$. We let $t_1^* = (1,0)$, $t_2^* = (x,0)$, $t_3^* = (0,1)$, $t_4^* = (0,x)$ such that $\{t_1^*,t_2^*,t_3^*,t_4^*\}$ is a k-linear base for $\mathsf{t}(\mathsf{Def}_M)$, and let $\alpha(t_i)$ be a 1-cocycle in the Yoneda complex $\mathsf{YC}^\bullet(L_\bullet,L_\bullet)$ that represents $t_i^* \in \mathsf{YH}^1(M,M) \cong \mathrm{Ext}^1_A(M,M)$. Note that a 1-cocyle $\phi \in \mathsf{YC}^1(L_\bullet,L_\bullet)$ is a pair (ϕ_0,ϕ_1) of A-linear maps $\phi_i : L_{i+1} \to L_i$ such that $d_0\phi_1 + \phi_0 d_1 = 0$ since $L_i = 0$ for $i > 2$. We may therefore choose

$$\alpha(t_1) = \left\{ \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot , \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \right\} \qquad \alpha(t_3) = \left\{ \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot , \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \right\}$$
$$\alpha(t_2) = \left\{ \begin{pmatrix} x & 0 \end{pmatrix} \cdot , \begin{pmatrix} 0 \\ -x \end{pmatrix} \cdot \right\} \qquad \alpha(t_4) = \left\{ \begin{pmatrix} 0 & x \end{pmatrix} \cdot , \begin{pmatrix} x \\ 0 \end{pmatrix} \cdot \right\}$$

Then the differential $d^{H_2} = (d_0^{H_2}, d_1^{H_2})$ is explicitly given by

$$d_0^{H_2} = d_0 + \sum_{1\le i\le 4} t_i\,\alpha(t_i)_0 = \begin{pmatrix} x^2+t_1+t_2x & y+t_3+t_4x \end{pmatrix} \cdot$$
$$d_1^{H_2} = d_1 + \sum_{1\le i\le 4} t_i\,\alpha(t_i)_1 = \begin{pmatrix} y+t_3+t_4x \\ -x^2-t_1-t_2x \end{pmatrix} \cdot$$

By construction, $d_0^{H_2} \circ d_1^{H_2} = 0$ in $H_2 \otimes_k \mathrm{Hom}_A(L_2,L_0)$ and we may check that this is the case:

$$\begin{aligned} d_0^{H_2} \circ d_1^{H_2} &= \begin{pmatrix} x^2+t_1+t_2x & y+t_3+t_4x \end{pmatrix} \cdot \begin{pmatrix} y+t_3+t_4x \\ -x^2-t_1-t_2x \end{pmatrix} \\ &= (x^2+t_1+t_2x)(y+t_3+t_4x) + (y+t_3+t_4x)(-x^2-t_1-t_2x) \\ &= [t_3,t_1] + [t_4,t_1]x + [t_3,t_2]x + [t_4,t_2]x^2 \end{aligned}$$

Since $(t_1,\dots,t_4)^2 = 0$ in H_2, this obstruction vansishes. The obstruction space $\mathrm{Ext}^2_A(M,M) \cong \mathsf{YH}^2(M,M) = k + kx$ has a k-linear base $\{s_1^* = 1, s_2^* = x\}$, and we can write the obstruction as

$$[t_3,t_1]s_1^* + ([t_4,t_1]+[t_3,t_2])s_2^* + [t_4,t_2]x^2 s_1^*$$

Since $s_1^*, s_2^* \neq 0$ while $x^2 s_1^* = 0$ in $\mathsf{YH}^2(M,M)$, it follows that $H_3 = k\langle\langle t_1,t_2\rangle\rangle/a_3$, where $a_3 = (f_1^2, f_2^2) + (t_1,t_2)^3$ and $f_1^2 = [t_3,t_1],\ f_2^2 = [t_4,t_1]+[t_3,t_2]$ are the second order approximations of f_1 and f_2. To lift ξ_2 to H_3, we choose $\alpha(t_4t_2)$ and $\alpha(t_2t_4)$ such that $d^1\alpha(t_4t_2) = -x^2s_1^*$ and $d^1\alpha(t_2t_4) = x^2s_1^*$, and find that

$$\begin{aligned} d^1\left(\left\{ (0\ 1)\cdot, \begin{pmatrix}0\\0\end{pmatrix}\cdot \right\}\right) = -x^2s_1^* \quad &\Rightarrow \quad \alpha(t_4t_2) = \left\{ (0\ 1)\cdot, \begin{pmatrix}0\\0\end{pmatrix}\cdot \right\} \\ &\Rightarrow \quad \alpha(t_2t_4) = \left\{ (0\ -1)\cdot, \begin{pmatrix}0\\0\end{pmatrix}\cdot \right\} \end{aligned}$$

Explicitly, the lifting ξ_3 is represented by the differential d^{H_3}, given by

$$\begin{aligned} d_0^{H_3} &= \begin{pmatrix} x^2+t_1+t_2x & y+t_3+t_4x+[t_4,t_2] \end{pmatrix}\cdot \\ d_1^{H_3} &= \begin{pmatrix} y+t_3+t_4x \\ -x^2-t_1-t_2x \end{pmatrix}\cdot \end{aligned}$$

Again, we compute the obstruction given by $d_0^{H_3}d_1^{H_3}$, and find that

$$\begin{aligned} d_0^{H_3}\circ d_1^{H_3} &= \begin{pmatrix} x^2+t_1+t_2x & y+t_3+t_4x+[t_4,t_2] \end{pmatrix}\cdot \begin{pmatrix} y+t_3+t_4x \\ -x^2-t_1-t_2x \end{pmatrix} \\ &= [t_3,t_1] + ([t_4,t_1]+[t_3,t_2])x - t_1[t_4,t_2] - t_2[t_4,t_2]x \\ &= ([t_3,t_1]-t_1[t_4,t_2])s_1^* + ([t_4,t_1]+[t_3,t_2]-t_2[t_4,t_2])s_2^* \end{aligned}$$

This implies that $H_4 = k\langle\langle t_1,t_2,t_3,t_4\rangle\rangle/a_4$, where $a_4 = (f_1^3, f_2^3) + (t_1,t_2,t_3,t_4)^4$ and $f_1^3 = [t_3,t_1]-t_1[t_4,t_2],\ f_2^3 = [t_4,t_1]+[t_3,t_2]-t_2[t_4,t_2]$ are the third order approximations of f_1 and f_2. We see that ξ_3 can be lifted to

$$H = k\langle\langle t_1,t_2,t_3,t_4\rangle\rangle/([t_3,t_1]-t_1[t_4,t_2], [t_4,t_1]+[t_3,t_2]-t_2[t_4,t_2])$$

and this implies that H is the pro-representing hull of the noncommutative deformation functor Def_M (with $f_1 = [t_3,t_1]-t_1[t_4,t_2]$ and $f_2 = [t_4,t_1]+[t_3,t_2]-t_2[t_4,t_2]$), and that Def_M is obstructed. The versal family $\xi \in \mathsf{Def}_M(H)$ is given by a lifting of complexes of $(L_\bullet, d_\bullet)$ to H. In concrete terms, the differential $d^H = (d_m^H)$ is given by

$$d_0^H = \begin{pmatrix} x^2+t_1+t_2x & y+t_3+t_4x+[t_4,t_2] \end{pmatrix}\cdot \quad \text{and} \quad d_1^H = \begin{pmatrix} y+t_3+t_4x \\ -x^2-t_1-t_2x \end{pmatrix}\cdot$$

and the versal family $\xi \in \mathsf{Def}_M(H)$ is the H-A bimodule $M_H = \operatorname{coker}(d_0^H)$.

5 Comparison with classical deformations

From the computations above, it follows that the classical deformation functor $\mathsf{Def}_M^{\mathsf{cl}}$ has a pro-representing hull $H^{\mathsf{cl}} = k[[t_1, \dots, t_4]]$ since $f_1^{\mathsf{cl}} = f_2^{\mathsf{cl}} = 0$, and that $\mathsf{Def}_M^{\mathsf{cl}}$ is unobstructed. Its versal family is given by the differential

$$d_0^H = \begin{pmatrix} x^2 + t_1 + t_2 x & y + t_3 + t_4 x \end{pmatrix} \cdot \quad \text{and} \quad d_1^H = \begin{pmatrix} y + t_3 + t_4 x \\ -x^2 - t_1 - t_2 x \end{pmatrix} \cdot$$

since $[t_4, t_2] = 0$ in H^{cl}. Hence the versal family is the H^{cl}-A bimodule $M_{H^{\mathsf{cl}}}$ given by

$$M_{H^{\mathsf{cl}}} = k[[t_1, \dots, t_4]][x,y]/(x^2 + t_1 + t_2 x,\ y + t_3 + t_4 x)$$

We see that there is an algebraization of H^{cl} and its versal family, given by the algebra $\mathfrak{H}^{\mathsf{cl}} = k[t_1, t_2, t_3, t_4]$ and the versal family

$$M_{\mathfrak{H}^{\mathsf{cl}}} = k[t_1, \dots, t_4][x,y]/(x^2 + t_1 + t_2 x,\ y + t_3 + t_4 x)$$

The corresponding family of classical deformations of M, parameterized by the closed points of $\operatorname{Spec} \mathfrak{H}^{\mathsf{cl}} = \mathbb{A}^4$, is $\{M_{\mathfrak{H}^{\mathsf{cl}}}(\tau) : \tau = (\tau_1, \dots, \tau_4) \in \mathbb{A}^4\}$ with

$$M_{\mathfrak{H}^{\mathsf{cl}}}(\tau) \cong k[x,y]/(x^2 + \tau_1 + \tau_2 x,\ y + \tau_3 + \tau_4 x)$$

This is a family of right A-modules of length 2.

References

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