$\mathbb{C}^e \supseteq V(f_1, \dots, f_m) =: X$ $\quad$ $(X, \mathcal{O}_X)$ analytiske rom

$f_i(0) = 0 \quad , i = 1, \dots, m$

$(X, 0) :=$ ekvivalensklasse

$$(X,0) \sim (X',0)$$

$\exists$ $U \ni 0$ åpen, $U \subseteq X$, $\quad U' \ni 0$ åpen, $U' \subseteq X'$

sterk topologi

s.a. $\quad U \cap X \underset{\text{biholomorf}}{\cong} U' \cap X'$

$$\mathcal{O}_{X,0} = \varinjlim_{\substack{U \ni 0 \\ \text{åpen}}} \mathcal{O}_X(U) = \mathbb{C}\{x_1, \dots, x_e\} / (f_1, \dots, f_m)$$

konvergente potensrekker

Antar

$\dim X = \text{krulldim}\ \mathcal{O}_{X,0} = 2$

$\mathcal{O}_{X,0}$ er normal

Def.

kompakt (projektiv) kurve

$$\begin{array}{ll} \tilde{X} & E = \pi^{-1}(0) \\ \downarrow \pi & \text{biholomorf på } \tilde{X} \setminus E \cong X \setminus \{0\} \\ X & \text{proper (projektiv)} \end{array}$$

resolusjon av $(X, 0)$

Oppgave

$X = V(f = xy - z^2) \subseteq \mathbb{C}^3$

Finn en resolusjon.

Løsning

$$\underset{\substack{\shortparallel \\ \widetilde{X}}}{\widetilde{\mathbb{C}^3} \cap \pi^{-1}(X) \setminus \pi^{-1}(0)} \subseteq \mathbb{C}^3_{x,y,z} \times \mathbb{P}^2(r,s,t)$$

$$\Big\downarrow \pi$$

$$X \subseteq \mathbb{C}^3$$

$$\operatorname{rk}\begin{bmatrix} r & x \\ s & y \\ t & z \end{bmatrix} \leq 1$$

U_r:

$r \neq 0$:

$$\operatorname{rk}\begin{bmatrix} 1 & x \\ \frac{s}{r} & y \\ \frac{t}{r} & z \end{bmatrix} \leq 1 \qquad \begin{aligned} y &= x\frac{s}{r} \\ z &= x\frac{t}{r} \end{aligned}$$

$$f = x \cdot x\frac{s}{r} - \left(x\frac{t}{r}\right)^2 = x^2\left(\frac{s}{r} - \left(\frac{t}{r}\right)^2\right)$$

$$U_r \cap \widetilde{X} = V\left(\frac{s}{r} - \left(\frac{t}{r}\right)^2\right)$$

$U_r \cong \mathbb{C}^2$
koordinater
$u_1 = x$, $v_1 = \frac{t}{r}$

U_s:

$s \neq 0$:

$$\operatorname{rk}\begin{bmatrix} \frac{r}{s} & x \\ 1 & y \\ \frac{t}{s} & z \end{bmatrix} \leq 1 \qquad \begin{aligned} x &= \frac{r}{s}y \\ z &= \frac{t}{s}y \end{aligned}$$

$$f = \frac{r}{s}y \cdot y - \left(\frac{t}{s}y\right)^2 = y^2\left(\frac{r}{s} - \left(\frac{t}{s}\right)^2\right)$$

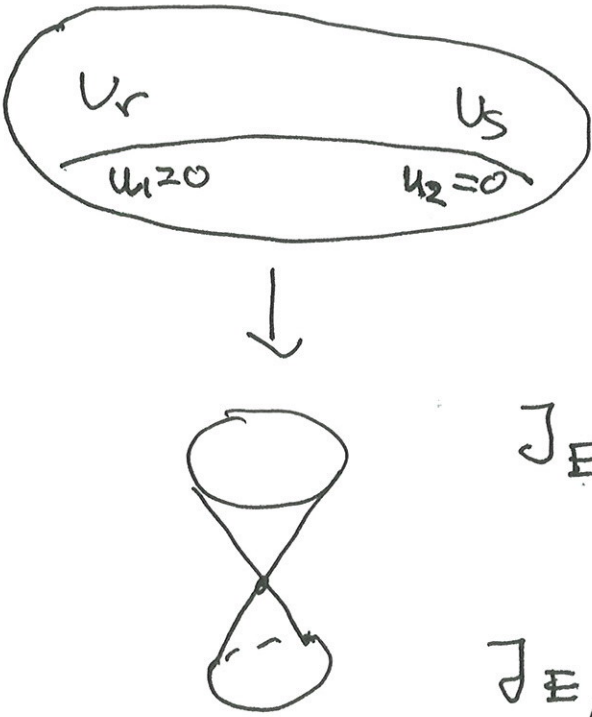
$U_s \cong \mathbb{C}^2$
koordinater
$u_2 = y$
$v_2 = \frac{t}{s}$

$U_r \cap U_s$:

$$u_1 = x = \frac{r}{s}y = \left(\frac{t}{s}\right)^2 y = v_2^{\,2} u_2 \Rightarrow u_1 = u_2 v_2^{\,2}$$

$$v_1 = \frac{t}{r} = \frac{\frac{t}{s}}{\frac{r}{s}} = \frac{v_2}{\left(\frac{t}{s}\right)^2} = \frac{v_2}{v_2^{\,2}} = \frac{1}{v_2} \Rightarrow v_1 = \frac{1}{v_2}$$

$\tilde{X} = U_r \cup U_s$



$E = \operatorname{Spec}\mathbb{C}[v_1] \cup \operatorname{Spec}\mathbb{C}[v_2]$

$v_1 = \frac{1}{v_2}$

$\cong \mathbb{P}^1$

$J_E =$ idealknippet til E i $\tilde{X}$

$$J_E / J_E^{\,2} \cong \mathcal{O}_{\mathbb{P}^1}(2)$$

$$E^2 := \deg_E \mathcal{H}om_{\mathcal{O}_E}\left(J_E/J_E^{\,2}, \mathcal{O}_E\right) = \underline{\underline{-2}}$$

Mer generelt

$E = \pi^{-1}(0) \subseteq \tilde{X}$

$= \bigcup E_i$ $\qquad$ E_i er irredusibel

$\tilde{X}$ er minimal dersom det ikke finnes $E_i \cong \mathbb{P}^1$ s.a. $E_i^2 = -1$

X ~~affin~~ Stein

$\iff H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$

Def.

X rasjonal (flate sing.) $\overset{\text{def}}{\iff}$ $R^i\pi_* \mathcal{O}_{\tilde{X}} = 0$ for $i > 0$

$$\begin{array}{c} \tilde{X} \\ \pi \downarrow \\ X \end{array}$$ en resolusjon

normal

Def.

$\tilde{X} \supseteq E = \bigcup E_i \longleftrightarrow$ dualgrafen $\Gamma(\tilde{X})$

$\bullet \; E_i$

$\underset{E_i}{\bullet} \!-\!\!-\!\!-\! \underset{E_j}{\bullet}$ $\qquad$ $E_i \cap E_j \neq \emptyset$

vekt$(E_i) = E_i^2$

Ex.

$X = V(xy - z^2)$

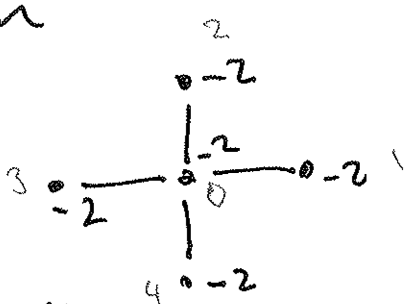
dualgraf: $\underset{-2}{\bullet}$

Teorem (Mumford)

Snittmatrisen $(E_i \cdot E_j)$ negativt <u>definitt</u>

Oppgave

Undersøk om



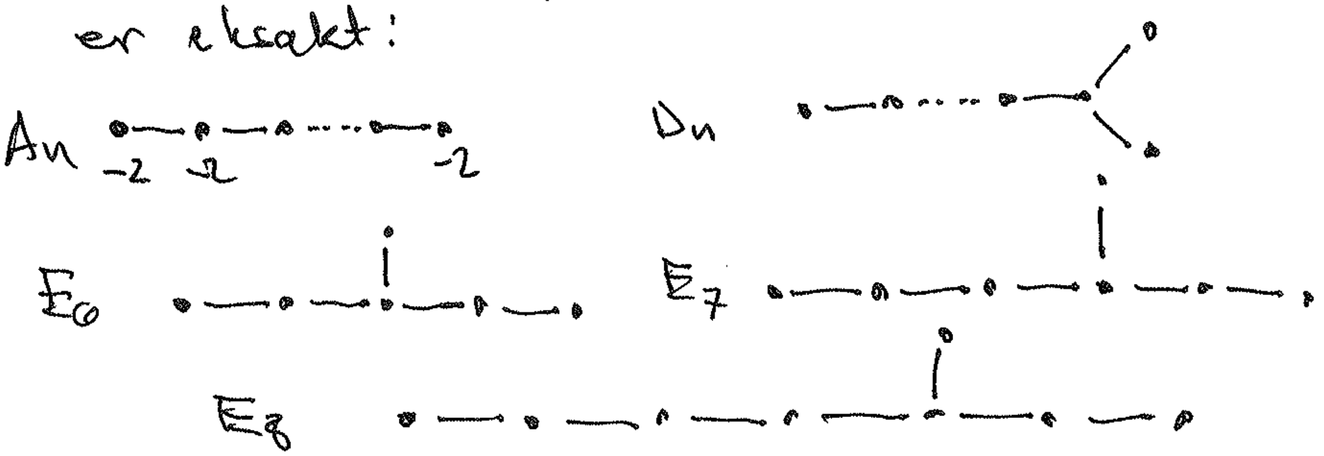
er grafen til en normal flatesing.
Hvis ikke modifiser vekter slik at ok.

Løsning

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix} \qquad \det = 0$$

Teorem

Grafene som gir (når alle vektene er -2) negativt definitt matrise er eksakt:



$$\tilde{X} \supseteq E = \cup E_i$$
$$\downarrow$$
$$X$$

Def

Fundamentalsykelen:

minste $Z = \sum_{r_i \geq 0} r_i E_i$ s.a. $Z \cdot E_i \leq 0 \ \forall E_i$

$$m_{\mathcal{O}_X} \mathcal{O}_{\tilde{X}} = \mathcal{O}(-Z) \quad \text{Artin}$$

Algoritme

$Z_0 = E$

Gitt Z_k:

i) $\exists \underset{E_i}{\overset{\shortparallel}{F}}$ med $Z_k \cdot F > 0$ så $Z_{k+1} = Z_k + F$

ii) $\nexists$ slik F så er $Z = Z_k$

$\boxed{\text{Eksempel}}$

$$Z_0 = E_0 + E_1 + E_2 + E_3$$

$$E_1 \cdot Z_0 = E_1 \cdot E_0 + E_1^2 = 1 - 2 = -1$$

$$E_0 \cdot Z_0 = -2 + 1 + 1 + 1 = 1$$

$$Z = Z_1 = Z_0 + E_0$$

*1

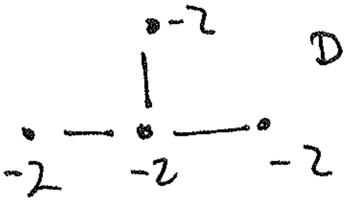
$\boxed{\text{Setning}}$

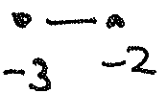
X rasjonal : $\underset{\substack{\| \\ e-1}}{m} = -Z^2$

$\boxed{\text{Oppgave}}$

Finn embeddingsdim. til de som er rasjonale:

D_4 $\quad \bullet \cong \mathbb{P}^1$

i)  A_1 ii) 

iii)  (iv) 

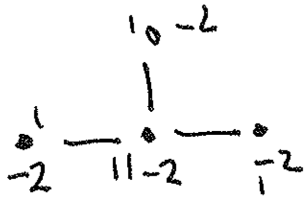
$\boxed{\text{Løsning}}$

i) $E = E_0$

$Z = E = E_0 \implies Z^2 = E_0^2 - 2$

$e - 1 = -(-2)$

$$\underline{\underline{e = 3}}$$

ii) 

$Z = 2E_0 + E_1 + E_2 + E_3$

$Z^2 = 2Z \cdot E_0 + Z \cdot E_1 + Z E_2 + Z E_3$

$= 2(-1) + 0 + 0 + 0$

$= -2$

$(2E_0 + E_1 + E_2 + E_3)(2E_0 + E_1 + E_2 + E_3) = 4E_0^2 + \ldots\ldots = \underline{\underline{-2}}$

Ann. scient. Éc. Norm. Sup.,
4e série, t. 10, 1977, p. 231 à 264.

EQUATIONS DEFINING RATIONAL SINGULARITIES

PAR JONATHAN M. WAHL

INTRODUCTION

Suppose $R = P/I$ is a complete two-dimensional rational singularity (e. g., [2]) of embedding dimension e, where P is a formal power series ring in e variables over an algebraically closed field k. The tangent cone $\overline{R} = \operatorname{gr} R$ is a quotient of the polynomial ring $\overline{P} = \operatorname{gr} P$.

THEOREM 1. (*see* 2.1). — *A minimal projective resolution for* $P/I = R$ *is*

$$0 \to P^{b_{e-2}} \xrightarrow{\varphi_{e-2}} \ldots \to P^{b_2} \xrightarrow{\varphi_2} P^{b_1} \xrightarrow{\varphi_1} P \to P/I \to 0,$$

where

(*a*) *the Betti numbers are* $b_i = i\binom{e-1}{i+1}$, $i \geqq 1$,

(*b*) *the associated graded sequence:*

$$0 \to \overline{P}^{b_{e-2}} \xrightarrow{\overline{\varphi}_{e-2}} \ldots \to \overline{P}^{b_2} \xrightarrow{\overline{\varphi}_2} \overline{P}^{b_1} \xrightarrow{\overline{\varphi}_1} \overline{P} \to \overline{P/I} \to 0,$$

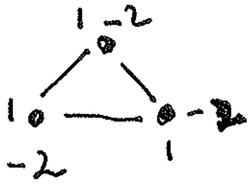
is a minimal projective resolution for $\overline{R}$, *and* $\overline{\varphi}_1$ *has degree* 2, $\overline{\varphi}_i$ *has degree* 1 $(i > 1)$.

We therefore may say that R is defined by quadratic equations, and all the higher syzygies are linear. The proof of this result is cohomological, but not difficult; one uses Castelnuovo's lemma on the projectivized tangent cone, showing it admits a 2-regular resolution, and then uses a variant of the Artin-Rees theorem to lift the equations for $\overline{R}$ to those for R (§ 1). Apparently, more elementary algebraic proofs are available using only that the multiplicity is one less than the embedding dimension (2.6).

The same techniques yield an analogous result for the "minimally elliptic" singularities of Laufer [12]; these are Gorenstein singularities (hence have self-dual resolutions), and include cones over elliptic curves and the cusp singularities of the two-dimensional Hilbert modular group.

THEOREM 2. (*see* 2.8). — *A minimally elliptic singularity* (*over* **C**) *with* $e \geqq 4$ *has a minimal resolution as in Theorem* 1, *except that*

(*a*) $$b_{e-2} = 1,\ b_i = \frac{i(e-i-2)}{e-1}\binom{e}{i+1}, \qquad i = 1, \ldots, e-3,$$

iii) 

? ? ?

$\boxed{\text{Oppgave}}$

Finn formen på den projektive resolusjonen når grafen er:

$\boxed{\text{Løsning}}$ $\qquad e = 4$

$$0 \longrightarrow P \longrightarrow P^3 \longrightarrow P \longrightarrow P/I \longrightarrow 0$$

$$b_1 = \binom{e-1}{1+1} = \binom{3}{2} = 3$$

$$b_2 = \binom{e-1}{3} = \binom{3}{3} = 1$$

$$P/I \cong \mathbb{C}[[\underset{\substack{\shortparallel \\ x}}{u^3}, \underset{\substack{\shortparallel \\ y}}{u^2v}, \underset{\substack{\shortparallel \\ z}}{uv^2}, \underset{\substack{\shortparallel \\ w}}{v^3}]]$$

$$\mathrm{rk}\begin{pmatrix} x & z & w \\ y & w & z \end{pmatrix} \leq 1$$

$$\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$