Obstructions to deforming space curves and non-reduced components of the Hilbert scheme

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§1 Introduction

Hilbert scheme

V: a projective variety over $k = \bar{k}$. $\mathrm{char}\, k = 0$
H: an ample divisor on V.

Notation

$$
\begin{array}{rcl}
\mathrm{Hilb}\, V & = & \text{the (full) Hilbert scheme of } V \\
\bigcup \text{ open} & & \\
\mathrm{Hilb}^{sc}\, V : & = & \{\text{smooth connected curves } C \subset V\} \\
\text{closed} \bigcup \text{ open} & & \\
\mathrm{Hilb}^{sc}_{d,g}\, V : & = & \{\text{curves of degree degree } d \text{ and genus } g\}
\end{array}
$$

$(d := (C \cdot H)_V)$

Hilbert scheme of space curves

$V = \mathbb{P}^3$: the projective 3-space over k
$C \subset \mathbb{P}^3$: a closed subscheme of $\dim = 1$

$d(C)$: degree of C (= $\sharp(C \cap H)$)
$g(C)$: genus of C (as a cpt. Riemann surf.)

We study the Hilbert scheme of space curves:

$$H^S_{d,g} := \mathrm{Hilb}^{sc}_{d,g}\,\mathbb{P}^3 = \left\{ C \subset \mathbb{P}^3 \;\middle|\; \begin{array}{c} \text{smooth and connected} \\ d(C) = d \text{ and } g(C) = g \end{array} \right\}$$

Why we study $H^S_{d,g}$?

Some reasons are:

- For every smooth curve C, there exists a curve $C' \subset \mathbb{P}^3$ s.t. $C' \simeq C$.
- $\mathrm{Hilb}^{sc}\,\mathbb{P}^3 = \bigsqcup_{d,g} H^S_{d,g}$
- More recently, the classification of the space curves has been applied to the study of bir. automorphism

$$\Phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$

(for the construction of Sarkisov links [Blanc-Lamy,2012]).

Some basic facts

- If $g \le d - 3$, then $H^S_{d,g}$ is irreducible [Ein,'86] and $H^S_{d,g}$ is generically smooth of expected dimension $4d$.
- In general, $H^S_{d,g}$ can become reducible, e.g
  $H^S_{9,10} = W_1^{(36)} \sqcup W_2^{(36)}$ [Noether].
- the Hilbert scheme of arith. Cohen-Macaulay (ACM, for short) curves are smooth [Ellingsrud, '75].

$$C \subset \mathbb{P}^3\text{: ACM} \overset{\text{def}}{\iff} H^1(\mathbb{P}^3, \mathcal{I}_C(l)) = 0 \text{ for all } l \in \mathbb{Z}$$

- $H^S_{d,g}$ can have many generically non-reduced irreducible components, e.g. [Mumford'62], [Kleppe'87], [Ellia'87], [Gruson-Peskine'82], etc.

Infinitesimal property of the Hilbert scheme

V: a projective variety over k
$C \subset V$: a subvariety of V
$\mathcal{I}_C$: the ideal sheaf defining C in V
$N_{C/V}$: the normal sheaf of C in V

Fact (Tangent space and Obstruction group)

1. The tangent space of $\mathbf{Hilb}\, V$ at $[C]$ is isomorphic to $\mathbf{Hom}(\mathcal{I}_C, \mathcal{O}_C) \simeq H^0(C, N_{C/V})$
2. Every obstruction $\mathbf{ob}$ to deforming C in V is contained in the group $\mathbf{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$. If C is a locally complete intersection in V, then $\mathbf{ob}$ is contained in $H^1(C, N_{C/V})$

$W \subset \mathbf{Hilb}\, V$: an irreducible closed subset of $\mathbf{Hilb}\, V$.
$[C] \in W$: a closed point of W
$C_\eta \in W$: the generic point of W

Definition

- We say C is unobstructed (resp. obstructed) (in V) if $\mathbf{Hilb}\, V$ is nonsingular (resp. singular) at $[C]$.
- We say $\mathbf{Hilb}\, V$ is generically smooth (resp. generically non-reduced) along W if $\mathbf{Hilb}\, V$ is nonsingular (resp. singular) at C_η.

Mumford's example (a pathology)

$S \subset \mathbb{P}^3$: a smooth cubic surface ($\simeq \mathbf{Blow}_{6\text{ pts}} \mathbb{P}^2$)
$h = S \cap \mathbb{P}^2$: a hyperplane section
E: a line on S
There exists a smooth connected curve

$$C \in |4h + 2E| \subset S \subset \mathbb{P}^3,$$

of degree $\mathbf{14}$ and genus $\mathbf{24}$.
Then C is parametrized by a locally closed subset

$$W = W^{(56)} \subset H^S_{14,24} \subset \mathbf{Hilb}^{sc} \mathbb{P}^3$$

of the Hilbert scheme.

The locally closed subset $W^{(56)}$ fits into the diagram

$$\begin{array}{ccccc} \overline{\left\{ C \subset \mathbb{P}^3 \;\middle|\; \begin{array}{c} C \subset {}^{\exists}S \text{ (smooth cubic)} \\ \text{and } C \sim 4h + 2E \end{array} \right\}} & =: & W^{(56)} & \subset & H^S_{14,24} \\ & & \downarrow \mathbb{P}^{39}\text{-bundle} & & \\ \begin{pmatrix} \text{family of smooth} \\ \text{cubic surfaces} \end{pmatrix} & =: & U & \underset{\text{open}}{\subset} & |\mathcal{O}_{\mathbb{P}^3}(3)| \simeq \mathbb{P}^{19}, \end{array}$$

where we have $\dim |\mathcal{O}_S(C)| = 39$ and $h^0(N_{C/\mathbb{P}^3}) = 57$.

$H^0(N_{C/\mathbb{P}^3})$ = the tangent space of $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ at $[C]$.
We have the following inequalities:

$$56 = \dim W \le \dim_{[C]} \mathrm{Hilb}^{sc}\,\mathbb{P}^3 \le h^0(N_{C/\mathbb{P}^3}) = 57.$$

Thus we have a dichotomy between (A) and (B):

- (A) $\overline{W}$ is an irred. comp. of $(\mathrm{Hilb}^{sc}\,\mathbb{P}^3)_{\mathrm{red}}$.
  $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ is generically non-reduced along $\overline{W}$.
- (B) There exists an irred. comp. $W' \supsetneqq W$.
  $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ is generically smooth along $\overline{W}$.

Which? $\rightsquigarrow$ The answer is (A). (It suffices to prove $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ is singular at the generic point $[C]$ of W. We will see later in §2)

History

Later many non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ were found by Kleppe['85], Ellia['87], Gruson-Peskine['82], Fløystad['93] and Nasu['05].
Moreover, to the question "How bad can the deformation space of an object be?", Vakil['06] has answered that

Law (Murphy's law in algebraic geometry)

Unless there is some a priori reason otherwise, the deformation space may be as bad as possible.

A naive question

Nowadays non-reduced components of Hilbert schemes are not seldom. However,

Question

What is/are the most important reason(s) (if any) for their existence?

Our answer is the following: (at least in Mumford's example,) a (-1)-curve E (i.e. $E \simeq \mathbb{P}^1$, $E^2 = -1$) on the (cubic) surface is the most important.

A generalization of Mumford's ex.

Theorem (Mukai-Nasu'09)

V: a smooth projective $\mathbf{3}$-fold. Suppose that

1. there exists a curve $E \simeq \mathbb{P}^1 \subset V$ s.t. $N_{E/V}$ is generated by global sections,
2. there exists a smooth surface S s.t. $E \subset S \subset V$, $(E^2)_S = -1$ and $H^1(N_{S/V}) = p_g(S) = 0$.

Then the Hilbert scheme $\mathbf{Hilb}^{sc}\, V$ has infinitely many generically non-reduced components.

In Mumford's ex., $V = \mathbb{P}^3$, S: a smooth cubic, E: a line.

Examples

We have many ex. of generically non-reduced components of $\mathbf{Hilb}^{sc} V$ for uniruled 3-folds V.

Ex.

1. Let V be a Fano 3-fold and let $-K_V = H + H'$, where H, H': ample. ${}^{\exists}S \in |H|$ (smooth). If $S \not\simeq \mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, then there exists a (-1)-$\mathbb{P}^1$ E on S.
2. Let $V \xrightarrow{\pi} F$ be a $\mathbb{P}^1$-bundle over a smooth surface F with $p_g(F) = 0$. Let S_1 be a section of π and A a sufficiently ample divisor on F. Then there exists a smooth surface $S \in |S_1 + \pi^* A|$. Take a fiber E of $S \to F$.

§2 Infinitesimal analysis of the Hilbert scheme

In the analysis of Mumford's ex., we develop some techniques to computing the obstruction to deforming a curve on a uniruled $\mathbf{3}$-fold ("obstructedness criterion").

Setting:
V: a uniruled $\mathbf{3}$-fold
S: a surface
E: (-1)-$\mathbb{P}^1$ on S
C: a curve on S
with $C \subset S \subset V$

$\xRightarrow{\text{Obst. Criterion}}$

Non-reduced components of $\mathbf{Hilb}^{sc}\, V$

Obstructions and Cup products

$\tilde{C} \subset V \times \operatorname{Spec} k[t]/(t^2)$:
a first order (infinitesimal) deformation of C in V
(i.e.,a tangent vector of $\operatorname{Hilb} V$ at $[C]$)

$$\begin{array}{ccc} \tilde{C} & \in & \{\text{1st ord. def. of } C\} \\ \updownarrow & & \updownarrow {}^{\exists}\text{1-to-1} \\ \alpha & \in & \operatorname{Hom}(\mathcal{I}_C, \mathcal{O}_C) \qquad (\simeq H^0(N_{C/V})) \end{array}$$

Define the cup product $\mathbf{ob}(\alpha)$ by

$$\mathbf{ob}(\alpha) := \alpha \cup \mathbf{e} \cup \alpha \in \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_C),$$

where $\mathbf{e} \in \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{I}_C)$ is the ext. class of an exact sequence $0 \to \mathcal{I}_C \to \mathcal{O}_V \to \mathcal{O}_C \to 0$.

Fact

A first order deformation $\tilde{C}$ lifts to a deformation over $\mathbf{Spec}\, k[t]/(t^3)$ if and only if $\mathbf{ob}(\alpha) = \mathbf{0}$.

Remark

- If $\mathbf{ob}(\alpha) \neq \mathbf{0}$, then $\mathbf{Hilb}\, V$ is singular at $[C]$.
- If C is a loc. complete intersection in V, then $\mathbf{ob}(\alpha)$ is contained in the small group $H^1(C, N_{C/V})$ ($\subset \mathbf{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$).

Exterior components

Let $C \subset S \subset V$ be a flag of a curve, a surface and a 3-fold (all smooth), and let $\pi_{C/S} : N_{C/V} \to N_{S/V}\big|_C$ be the natural projection.

Definition

Define the *exterior component* of α and $\mathbf{ob}(\alpha)$ by

$$\begin{aligned} \pi_S(\alpha) &:= H^0(\pi_{C/S})(\alpha) \\ \mathbf{ob}_S(\alpha) &:= H^1(\pi_{C/S})(\mathbf{ob}(\alpha)), \end{aligned}$$

respectively.

Infinitesimal deformation with pole

Let $E \subset S \subset V$ be a flag of a curve, a surface and a 3-fold.

Definition

A rational section v of $N_{S/V}$ admitting a pole along E, i.e.

$$v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V}),$$

is called an infinitesimal deformation with a pole.

Remark (an interpretation)

Every inf. def. with a pole induces a 1st ord. def. of the open surface $S^\circ = S \setminus E$ in $V^\circ = V \setminus E$ by the map

$$H^0(N_{S/V}(E)) \hookrightarrow H^0(N_{S^\circ/V^\circ})$$

Obstructedness Criterion

Now we are ready to give a sufficient condition for a first order infinitesimal deformation of $\tilde{C}$ $(\subset V \times \operatorname{Spec} k[t]/(t^2))$ of C in V to a second order deformation $\tilde{\tilde{C}}$ $(\subset V \times \operatorname{Spec} k[t]/(t^3))$.

Condition (☆)

We consider $\alpha \in H^0(N_{C/V})$ satisfying the following condition (☆): the ext. comp. $\pi_S(\alpha)$ of α lifts to an inf. def. with a pole along E, say v, and its restriction $v\big|_E$ to E does not belong to the image of the map $\pi_{E/S}(E) := \pi_{E/S} \otimes \mathcal{O}_S(E)$.

$$
\begin{array}{ccccccc}
H^0(N_{C/V}) & \ni \alpha & & & & & H^0(N_{E/V}(E)) \\
\downarrow \pi_{C/S} & \downarrow & & & & & \downarrow \pi_{E/S}(E) \\
H^0(N_{S/V}\big|_C) & \ni \pi_S(\alpha) & \stackrel{\text{res.}}{\longleftarrow} & v & \stackrel{\text{res.}}{\longmapsto} & v\big|_E \in & H^0(N_{S/V}(E)\big|_E) \\
& (=v\big|_C) & & & & & \\
\cap & & & \Cap & & & \\
H^0(N_{S/V}(E)\big|_C) & \stackrel{\text{res.}}{\longleftarrow} & & H^0(N_{S/V}(E)) & & &
\end{array}
$$

Theorem (Mukai-Nasu'09)

Let $C, E \subset S \subset V$ be as above. Suppose that $E^2 < 0$ on S, and let $\alpha \in H^0(N_{C/V})$ satisfy (☆). If moreover,

1. Let $\Delta := C + K_V\big|_S - 2E$ on S. Then
$$(\Delta \cdot E)_S = 2(-E^2 + g(E) - 1) \qquad (2.1)$$
2. the res. map $H^0(S, \Delta) \to H^0(E, \Delta\big|_E)$ is surjective,

then we have $\mathbf{ob}_S(\alpha) \neq 0$.

Remark

If E is a (-1)-$\mathbb{P}^1$ on S, then the RHS of (2.1) is equal to 0.

How to apply Obstructedness Criterion

(Mumford's ex. $V = \mathbb{P}^3$)
Every general member $C \subset \mathbb{P}^3$ of Mumford's ex. $W^{(56)} \subset \mathbf{Hilb}^{sc} \mathbb{P}^3$ is contained in a smooth cubic surface S and $C \sim 4h + 2E$ on S (E: a line, h: a hyp. sect.).
Let t_W denote the tangent space of W at $[C]$ ($\dim t_W = \dim W = 56$).
Then there exists a first order deformation

$$\tilde{C} \longleftrightarrow \alpha \in H^0(C, N_{C/\mathbb{P}^3}) \setminus t_W.$$

of C in $\mathbb{P}^3$.

Claim

$\mathrm{ob}(\alpha) \neq 0$.

Proof.

Since $H^1(N_{S/\mathbb{P}^3}(E-C))=0$, the ext. comp. $\pi_{C/S}(\alpha)\in H^0(N_{S/\mathbb{P}^3}\big|_C)$ of α has a lifts to a rational section $v\in H^0(N_{S/\mathbb{P}^3}(E))$ on S (an inf. def. with a pole). By the key lemma below, the restriction $v\big|_E$ to E is not contained $\operatorname{im}\pi_{E/S}(E)$. Since $C\sim 4h+2E=-K_{\mathbb{P}^3}\big|_S+2E$, the divisor Δ is zero. Thus the condition (1) and (2) are both satisfied. □

Lemma (Key Lemma)

Since C is general, the finite scheme $Z:=C\cap E$ of length 2 is not cut out by any conic in $|h-E|\simeq\mathbb{P}^1$ on S.

§3 Application to Kleppe's conjecture

Minimal degree $s(W)$ for $W \subset H^S_{d,g}$

$\mathrm{Hilb}^{sc}\,\mathbb{P}^3$: the Hilb. sch. of sm. con. curves $C \subset \mathbb{P}^3$
$H^S_{d,g} \subset \mathrm{Hilb}^{sc}\,\mathbb{P}^3$: the subsch. of curves of deg. d and gen. g
$W \subset H^S_{d,g}$: an irreducible closed subset
$C \in W$: a general member of W

Definition (minimal degree of W)

$$s(W) := \min\left\{s \in \mathbb{N} \;\middle|\; H^0(\mathbb{P}^3, \mathcal{I}_C(s)) \neq 0\right\}$$

= the minimal degree of a surface $S \supset C$

s-maximal subsets of $H^S_{d,g}$

Definition (Kleppe'87)

$W \subset H^S_{d,g}$ is called $s(W)$-maximal if it is maximal w.r.t $s(W)$.

W: s-maximal $\Longrightarrow$ $s(V) > s$ for any closed irreducible subset $V \supsetneqq W$.

Ex. (Mumford's ex.)

$$W = \left\{ C \subset \mathbb{P}^3 \mid C \subset {}^{\exists}S \text{ (sm. cubic) and } C \sim 4h + 2E \right\}^{-}$$

is a 3-maximal subset of $H^S_{14,24}$.

First properties of s-maximal subsets

In what follows, we assume that

1. $W \subset H^S_{d,g}$: a s-maximal subset
2. a general member $C \subset S$, where S: a smooth surface of deg s.

Proposition

Suppose that $s \leq 4$ and $d > s^2$. Then

1. If C is not a c.i. when $s = 4$, then
$\dim W = (4 - s)d + g + \binom{s+3}{s} - 2$
2. If $H^1(\mathbb{P}^3, \mathcal{I}_C(s)) = 0$ and if C is not a c.i. when $s = 4$, then W is a generically smooth component of $H^S_{d,g}$.

3-maximal subsets of $H^S_{d,g}$

Let $s = 3$. If $d > 3^2 = 9$, then $\dim W = d + g + 18$.

Fact

Every irreducible component of $H^S_{d,g}$ is of dimension greater than or equal to $4d$ $(= \chi(N_{C/\mathbb{P}^3}))$

Thus if W is a component, then

$$\dim W \geq 4d \iff g \geq 3d - 18.$$

Conjecture

Conjecture (Kleppe'87, a ver. modified by Ellia)

Let $d > 9$, $g \geq 3d - 18$, and let $W \subset H^S_{d,g}$ be a 3-maximal subset. If a general member C of W satisfies

1. $H^1(\mathbb{P}^3, \mathcal{I}_C(3)) \neq 0$, and
2. $H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0$

then W is a gen. non-reduced irred. component of $H^S_{d,g}$.

Remark

1. The linearly normality assumption ($H^1(\mathcal{I}_C(1)) = 0$) was missing in the original ver. of the conjecture. (pointed out by Ellia['87] with a counterexample).
2. The tangential dimension $h^0(N_{C/\mathbb{P}^3})$ of $H^S_{d,g}$ at $[C]$ is greater than $\dim W$ by $h^1(\mathcal{I}_C(3))$.
3. The subset W can be described more explicitly in terms of the coordinate $(a; b_1, \ldots, b_6)$ of C in $\operatorname{Pic} S \simeq \mathbb{Z}^7$

Known results

In the following cases, Kleppe's conjecture is known to be true.

1. $g > 7 + \dfrac{(d-2)^2}{8}$ and $d \geq 18$ [Kleppe'87]
2. $d \geq 21$ and $g > G(d,5)$ [Ellia'87] [1]
3. $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 1$ [Nasu'05] [2]

[1] $G(d,5)$ denotes the max. genus of curves of degree d, not contained in any quartic surface. $G(d,5) \approx d^2/10$ for $d >> 0$.

[2] proved by computing cup products

Let $d > 9$ and $g \geq 3d - 18$ and let $W \subset H^S_{d,g}$ and C as in Kleppe's conjecture.

Lemma

The following conditions are equivalent:

1. C is quadratically normal, i.e., $H^1(\mathbb{P}^3, \mathcal{I}_C(2)) = 0$.
2. $(C \cdot E) \geq 2$ for every line E on S
3. Let $C \sim (a; b_1, \ldots, b_6)$ with some basis of $\mathrm{Pic}\, S \simeq \mathbb{Z}^7$. Then $b_i \geq 2$ for all $i = 1, \cdots, 6$.
4. Let $h \in \mathrm{Pic}\, S$ be the cls. of hyp. sections. The base locus of the complete lin. sys. $|C - 3h|$ contains no double lines $2E_i$, and no triple lines $3E_i$.

Main Theorem

Theorem (—'09)

Kleppe's conjecture is true if C is quadratically normal, i.e.,

$$H^1(\mathbb{P}^3, \mathcal{I}_C(2)) = 0.$$

How to prove Main Theorem

As we see in the Mumford's ex., it suffices to prove that $\mathbf{ob}(\alpha) \neq \mathbf{0}$ for every

$$\alpha \in H^0(C, N_{C/\mathbb{P}^3}) \setminus t_W,$$

where t_W is the tangnet space of W at $[C]$.
Note that the ext. comp. $\pi_S(\alpha)$ of α lifts to a rational section

$$v \in H^0(S, N_{S/\mathbb{P}^3}(F)) \setminus H^0(S, N_{S/\mathbb{P}^3}),$$

where F is the fixed component of the lin. sys. $|C - 3h|$.
Then we apply the obstructedness criterion for a first order deformation $\tilde{C}$ ($\longleftrightarrow \alpha$) of C in $\mathbb{P}^3$.

A more recent result

Another progress has been made:

Theorem (Kleppe'12)

Kleppe's conjecture is true provided that;

1. $b_6 = 2$, $b_5 \geq 4$, $d \geq 21$ and $(a; b_1, \ldots, b_6) \neq (\lambda + 12, \lambda + 4, 4, \ldots, 4, 2)$, ${}^\forall \lambda \geq 2$,
2. $b_6 = 1$, $b_5 \geq 6$, $d \geq 35$ and $(a; b_1, \ldots, b_6) \neq (\lambda + 18, \lambda + 6, 6, \ldots, 6, 1)$, ${}^\forall \lambda \geq 2$,
3. $b_6 = 1$, $b_5 = 5$, $b_4 \geq 7$, $d \geq 35$ and $(a; b_1, \ldots, b_6) \neq (\lambda + 21, \lambda + 7, 7, \ldots, 7, 5, 1)$, ${}^\forall \lambda \geq 2$.

§4 Obstruction to deforming curves on a quartic surface

Quartic surfaces containing a line

Similarly, we can compute the obstructions to deforming curves on a smooth quartic surface.

Assume that:
$S \subset \mathbb{P}^3$: a smooth quartic surface (a K3 surface),
E: a line on S,
F $(\sim h - E)$: a plane cubic curve cut out by a plane $H \supset E$,
$\operatorname{Pic} S \simeq \mathbb{Z}E \oplus \mathbb{Z}F$, and the intersection matrix is given by

$$\begin{pmatrix} E^2 & E \cdot F \\ E \cdot F & F^2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix}.$$

Then every curve C on S is expressed in $\mathrm{Pic}\, S$ by

$$C \sim aE + bF \qquad (a, b \geq 0)$$

with $d = a + 3b$ and $g = 3ab - b^2 + 1$.
Let W be a 4-maximal subset containing $[C]$. If $d > 16$ and $a \neq b$, then

$$\dim W = g + 33.$$

Moreover, we see that....

Theorem (Kleppe'12 (with Ottem))

Suppose that $d > 16$ and $4 < a < b$. Then

1. If $3b - 2a \geq 3$, then $h^1(\mathcal{I}_C(4)) = 0$. In particular, W is a generically smooth component of $H^S_{d,g}$.
2. If $3b - 2a \leq 2$, then $h^1(\mathcal{I}_C(4)) \neq 0$. Moreover, W is a generically non-reduced component of $H^S_{d,g}$.

In fact, we see that

$$h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = \begin{cases} 1 & (3b - 2a = 2) \\ 2 & (3b - 2a = 1) \\ 4 & (3b - 2a = 0) \end{cases}$$

By computing cup products, we have proved the following:

Theorem

Let C and W be as in the thm. If

$$3b - 2a = 2 \quad (\Rightarrow h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = 1),$$

then there exists a first order deformation $\tilde{C}$ of C in $\mathbb{P}^3$ which does not lift to a deformation over $\mathbf{Spec}\, k[t]/(t^3)$.

However, for the other case (where $3b - 2a = 1, 0$) we have not yet proved the obstructedness of a general $C \in W$.

Quartic surfaces containing a conic

We have many variations of a smooth quartic surface S containing $E \simeq \mathbb{P}^1$, e.g., the one containing conics E_1, E_2.

Assume that:
$S \subset \mathbb{P}^3$: a smooth quartic surface (a K3 surface),
E_1, E_2: conics on S such that $h \sim E_1 + E_2$,
$\mathrm{Pic}\, S \simeq \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$, and the intersection matrix is given by

$$\begin{pmatrix} E_1^2 & E_1 \cdot E_2 \\ E_1 \cdot E_2 & E_2^2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 4 & -2 \end{pmatrix}$$

Then every curve C on S is expressed in $\operatorname{Pic} S$ by

$$C \sim aE + bF \qquad (a, b \geq 0)$$

with $d = 2a + 2b$ and $g = 4ab - a^2 - b^2 + 1$.
Let W be a 4-maximal subset containing $[C]$. If $d > 16$ and $a \neq b$, then

$$\dim W = g + 33.$$

Moreover, we see that....

Theorem (Kleppe'12 (with Ottem))

Suppose that $d > 16$ and $a \neq b > 4$. If

$$\frac{b+4}{2} \leq a \leq 2b - 4,$$

then $h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = 0$. In particular, W is a generically smooth component of $H^S_{d,g}$.

Otherwise, we see that

$$h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = \begin{cases} 1 & (2b - a = 3) \\ 4 & (2b - a = 2) \\ 9 & (2b - a = 1) \\ 16 & (2b - a = 0) \end{cases}$$

By computing cup products, we have proved the following:

Theorem

Let C be a general member of W, and suppose that

$$2b - a = 3 \quad (\Rightarrow h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = 1).$$

Then there exists a first order deformation $\tilde{C}$ of C in $\mathbb{P}^3$ which does not lift to a deformation over $\mathbf{Spec}\, k[t]/(t^3)$.

However, for the other case (where $2b - a = 2, 1, 0$) we have not yet proved the obstructedness of a general $C \in W$.

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