Obstructions to deforming space curves and non-reduced components of the Hilbert scheme

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Curves embedded into the projective 3-space $\mathbb{P}^3$ are called *space curves*. The problem of classifying space curves is classical and nowadays the problem is well understood (at least, theoretically) in terms of the Hilbert scheme. Let $H^S_{d,g}$ denote the Hilbert scheme of smooth connected curves $C \subset \mathbb{P}^3$ of degree d and genus g. For the classification, it suffices to determine the all irreducible components of $H^S_{d,g}$ for a given degree d and a given genus g. However this is not easy task in general.

On the other hand, among all space curves, the curves which are contained in a (smooth) surface $S \subset \mathbb{P}^3$ of low-degree $s \leq 4$ have been intensively studied by many authors, e.g. Kleppe[3], Ellia[1], Gruson-Peskine[2], etc. In particular, Kleppe studied maximal families of smooth connected curves lying on a smooth cubic surface, and gave a conjecture concerning generically non-reduced irreducible components of the Hilbert scheme $H^S_{d,g}$. In this talk, I give a proof of some particular case of his conjecture. It is known that the obstruction $\mathrm{ob}(\varphi)$ to lifting a first order deformation $\tilde{C}$ ($\leftrightarrow \varphi \in H^0(C, \mathcal{N}_{C/\mathbb{P}^3})$) of a curve C in $\mathbb{P}^3$ to the second order deformations is given by the cup product

$$H^0(C, \mathcal{N}_{C/\mathbb{P}^3}) \times H^0(C, \mathcal{N}_{C/\mathbb{P}^3}) \longrightarrow H^1(C, \mathcal{N}_{C/\mathbb{P}^3}), \qquad \varphi \longmapsto \varphi \cup \varphi = \mathrm{ob}(\varphi),$$

where $\mathcal{N}_{C/\mathbb{P}^3}$ is the normal bundle of C in $\mathbb{P}^3$. For the proof, we compute this cup product and prove that it is nonzero by using the technique developed in [4].

More recently, in the joint paper [5] with Ottem, Kleppe has studied maximal families of the Hilbert scheme $H^S_{d,g}$ of the space curves whose general member is contained in a smooth quartic surface (i.e. a K3 surface) of Picard number 2, and has systematically constructed many examples of generically non-reduced irreducible components of $H^S_{d,g}$. If the time allows, I will discuss the obstruction to deforming such space curves (i.e. curves lying on a smooth quartic surface).

References

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