Deformations of determinantal schemes and modules of max grade

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3.03.2010
HiO

Main result, set-up and history.

Let $k = \bar{k}$, $R = k[x_0, \dots, x_n]$ poly. ring

Given $t \times (t+c-1)$ matrix α, $t \geq 2$, $c \geq 2$, s.t

$$\varphi : \underbrace{\bigoplus_{i=1}^{t} R(b_i)}_{= F} \xrightarrow{\alpha^T} \underbrace{\bigoplus_{j=0}^{t+c-2} R(a_j)}_{= G},$$

Let $\boxed{M = \operatorname{coker} \varphi^*}$, $\varphi^* : G^* \to F^*$

Let $\boxed{I = I_t(\varphi) = \text{ideal of max minors}}$

$\boxed{\text{Def}}$ Put $A := R/I$

a) Then $X := \operatorname{Proj}(A)$ (resp. A) is <u>standard determ.</u> scheme (resp. algebra)

$\iff \operatorname{codim}_R A = c$ i.e. the largest possible

b) <u>good determ.</u> $\iff$ <u>standard determ.</u> and <u>generic complete intersection</u>

c) M has <u>max grade</u> $\iff$ A is stand. determ.

(in which case $I = \operatorname{ann}(M)$)

Thm (Hochster, Eagon-Northcott)

A is CM (provided A is stand. determ.)

Indeed we have an induced map

$$G^* \otimes F \to R$$

and more generally an R-free res. (Eagon-Northcott)

EN
$$0 \to \Lambda^{t+c-1} G^* \otimes S_{c-1}(F) \otimes \Lambda^t F \to \Lambda^{t+c-2} G^* \otimes S_{c-2}(F) \otimes \Lambda^t F \to \cdots \to \Lambda^t G^* \otimes S_0(F) \otimes \Lambda^t F \to R \to A \to 0$$

Hence $\mathrm{pd} A = c$ (the codim) $\Rightarrow$ $\boxed{A \text{ is CM}}$

Let $\boxed{B \text{ be cokernel of } F \to G}$ $\Rightarrow$

$$0 \to \mathrm{Hom}(B, R) \to G^* \to F^* \to M \to 0$$

The following R-free resolution (Buchsbaum-Rim) is a (min.) res. of $\mathrm{Hom}(B,R)$ and hence of M:

BR
$$0 \to \Lambda^{t+c-1} G^* \otimes S_{c-2}(F) \otimes \Lambda^t F \to \Lambda^{t+c-2} G^* \otimes S_{c-3}(F) \otimes \Lambda^t F \to \cdots \to \Lambda^{t+1} G^* \otimes S_0(F) \otimes \Lambda^t F \to G^* \to F^* \to M \to 0$$

$\Rightarrow$ M is a max. CM A-module

DEF

$$\boxed{K_c := \dim_k {}_0\mathrm{Hom}(B, R(a_{t+c-2}))}$$

Suppose

$$\boxed{a_0 \le a_1 \le a_2 \le \cdots \quad \text{and} \quad b_1 \le b_2 \le \cdots \quad b_t}$$

Consider

$$G^* = \oplus R(-a_j) \xrightarrow{\alpha} F^* = \oplus R(-b_i)$$

and the

<u>degree matrix</u>, $\mathfrak{U}$, of α = $\begin{bmatrix} a_0-b_1 & a_1-b_1 & a_2-b_1 & \cdots \\ a_0-b_2 & a_1-b_2 & & \cdots \\ \vdots & \vdots & & \\ a_0-b_t & a_1-b_t & & \cdots \end{bmatrix}$

<u>Remark</u>

a) Determ. schemes with fixed $\mathfrak{U}$ have the same Hilbert poly. $p(t)$

b) Suppose α is min $(a_j - b_i = 0 \Rightarrow \alpha_{ij} = 0)$, then we have by (EN)

$\exists$ stand. determ. $\Longleftrightarrow$ $\exists$ good determ. $\Leftrightarrow$ $a_{i-1} > b_i$ $\forall i = 1, 2, \ldots, t$

(otherwise "too many zero's in α)

<u>Def.</u> Let

$$\boxed{W(\underline{b};\underline{a}) \subseteq \mathrm{Hilb}^{p(t)}(\mathbb{P}^n)}$$

be the <u>locus</u> of <u>all</u> good <u>determ. schemes</u> with fixed $\mathfrak{U}$.

<u>Remark</u> $\overline{W(\underline{b},\underline{a})}$ is irreducible

Indeed let $\mathbb{V} = \mathbb{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{G})$ be the affine scheme $\mathbb{A}^N$ parametrizing all morph. $\mathcal{F} \to \mathcal{G}$,

$$N = \sum \binom{a_j - b_i + m}{m} =: \hom(\mathcal{F}, \mathcal{G})$$

Since $\exists$ open $\mathbb{U} \subseteq \mathbb{V}$ and a dominating morphism

$$\mathbb{U} \longrightarrow W(\underline{b}; \underline{a}),$$

it follows that $W(\underline{b}; \underline{a})$ is **irreducible** OK

Continuing this argument, we get

Proposition [KMMNP, 01] and [KM, 05]

$$\dim W(\underline{b}; \underline{a}) \leq \hom(\mathcal{F}, \mathcal{G}) - \operatorname{aut}(\mathcal{F}) - \operatorname{aut}(\mathcal{G}) + \hom(\mathcal{G}, \mathcal{F}) + \operatorname{aut}(\mathcal{B})$$

where $\operatorname{aut}(\mathcal{M}) = \hom(\mathcal{M}, \mathcal{M})$

$\Longrightarrow$

$$\boxed{\dim W(\underline{b}; \underline{a}) \leq \lambda_c + K_3 + K_4 + \cdots + K_c}$$ where

$$\lambda_c = \sum \binom{a_j - b_i + m}{m} - \sum \binom{a_j - a_i + n}{n} - \sum \binom{b_j - b_i + n}{n} + \sum \binom{b_i - a_j + n}{n} + 1$$

Conjecture [KM, 05] Suppose $a_{i-2} \geq b_i$ for all $i = 1, 2, \ldots, t$ (or something slightly stronger)

$$\boxed{\text{Equality always holds}}$$

(also for $n - c = 0$, i.e. for zero-dim schemes some explicit given counterexample [KM, 09])

<u>Theorem 1</u> [K,10] Equality always holds for $n-c \geq 1$.

<u>Remark</u>

$c = 2 \Rightarrow \dim W(\underline{b};\underline{a}) = \lambda_2$ by [Ellingsrud, 75]

$c = 3 \Rightarrow \dim W(\underline{b};\underline{a}) = \lambda_3 + K_3$ is mostly [KMMNP, 01]

$4 \leq c \leq 5$ is proved in [KM, 05]

(supposing char $k = 0$ if $c = 5$)

If $c \geq 5$ and $a_{t+3} > a_{t-2}$ (and $a_0 > b_t$), then equality always holds, also for $n-c=0$ [KM, 09]

<u>Problem</u>

Is $\overline{W(\underline{b};\underline{a})}$ an irreducible component of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$?

Is $W(\underline{b};\underline{a})$ generically smooth ? (i.e. Hilb^p smooth along some open $\subseteq W(\underline{b},\underline{a})$)

<u>Theorem 2</u> [K, 10]. Suppose $a_{i-\min(3,t)} \geq b_i \quad \forall i$

Then $\overline{W(\underline{b},\underline{a})}$ gen. smooth irred. comp of $\mathrm{Hilb}(\mathbb{P}^n)$

<u>Remark</u> Thm 2 is known for

$c = 2$ by [Ellingsrud, 75], and

1. True also for $n-c=1$
2. — " — that any determ. scheme is unobst.

$c = 3$ by [KMMNP, 01]. Both 1 and 2 are false

$c = 4$ by [KM, 05]

$c \geq 3$ and $a_{t+1} > a_t + a_{t-1} - b_1$ (and $a_0 > b_t$) by [KM, 09]. 1 OK

In [KM, 09] we conjecture Thm 2 provided $a_0 > b_t$

We just look to $\boxed{n-c \geq 1}$

<u>Example</u> $\overline{W(\underline{b},\underline{a})}$ not irred. comp of Hilb for every $c \geq 3$

$$\overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 2 \\ 1 & 1 & \cdots & 1 & 2 \end{bmatrix}}^{c} \quad \text{in } \mathbb{P}^{c+1}, \text{ i.e. } \dim X = 1$$

$\boxed{c=3} \Rightarrow X$ is a determ. curve in $\mathbb{P}^4$ of $(d,g) = (7,3)$

$$\dim_{(X)} \mathrm{Hilb}(\mathbb{P}^4) \geq \chi(\mathcal{N}_X) = 5d+1-g = \underline{\underline{33}}$$

$$\dim W(\underline{0}; 1,1,1,2) \leq \lambda_3 + K_3 = 32+0 = \underline{\underline{32}}$$

$\boxed{c=4}$ curve in $\mathbb{P}^5$, $(d,g) = (9,4)$

$$h^0(\mathcal{N}_X) - h^1(\mathcal{N}_X) = 6d + 2(1-g) = \underline{\underline{48}}$$

$$\dim W(0,0; 1,1,1,1,2) \leq 8\cdot 6 + 2\cdot 21 - (16+4\cdot 6+1) - 4 + 1 = \underline{\underline{46}}$$

etc

<u>Problem</u> Why do we sometimes have

$$\overline{W(\underline{b};\underline{a})} \subsetneq V, \quad V \text{ irred comp. of Hilb?}$$

<u>Related problem</u>

Are deformations (liftings) of a determinantal always determinantal?

<u>Thm 2</u> really proves Yes in a precise way

(Indeed the answer is Yes for every good determ scheme satisfying ${}_0\mathrm{Ext}^i_A(M,M) = 0$ for $i = 1,2$)

and hence we get $\overline{W(\underline{b},\underline{a})} = V$, as well as

gen. smoothness.