Commuting nilpotent matrices and Artin algebras.

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Abstract

Fix an $n \times n$ nilpotent matrix B whose Jordan blocks are given by the partition P of n. Assume the field k is closed. Consider the irreducible variety N_B parametrizing nilpotent $n \times n$ matrices A that commute with B. What partition $Q(P)$ occurs for a generic A?

The ring $k[A, B]$ is an Artinian ring. V. Baranovsky, R. Basili, A. Premet and others explored the connection between fhe family $P(n)$ of pairs of commuting nilpotent matrices and the Hilbert scheme parametriizing Artin algebra quotients of $k[x, y]$. P. Oblak and T Košir showed that when A is generic, then $k[A, B]$ is Gorenstein. However, the Hilbert funciion of this ring determines $Q(P)$. A result of F.H.S. Macaulay then shows that $Q(P)$ has parts that differ pairwise by at least two. P. Oblak has determined the largest part of $Q(P)$. We report on these results and others oonnecting the study of Artinian algebras and commuting nilpotent matrices. In work joint with R. Basili and L. Khatami, we give a criterion on A for $k[A, B]$ to be Gorenstein.

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1 When do two matrices commute?

We consider the space $Mat_n(\mathbf{C})$ of $n \times n$ matrices B with complex entries (in $\mathbf{C}$) and the vector space $V = \mathbf{C}^n$. So B is the matrix in the standard basis of a map $m_B : V \to V$. When B has distinct eigenvalues, then A commutes with B iff A, B are "simultaneously diagonalizable". We can see this in a special case, as follows. Suppose that $n = 3$ and B is diagonal:

$$B = \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}.$$

It is easy to see that for $A = (b_{ij}), 1 \leq i, j \leq 3$

$$BA - AB = \begin{pmatrix} 0 & (d-e)b_{12} & (d-f)b_{13} \\ (e-d)b_{21} & 0 & (e-f)b_{23} \\ (f-d)b_{31} & (f-e)b_{32} & 0 \end{pmatrix}$$

So when d, e, f are distinct, $A \in \mathcal{C}(B)$, the centralizer of B, only if $b_{ij} = 0$ for $i \neq j$: when A is diagonal.

Another way to write this is:

Lemma. Let B have n-distinct eigenvalues. Then $A \in \mathcal{C}(B)$ iff TFAE:

i. B, A are simultaneously diagonalizable.

ii. A is a polynomial in B.

Proof. (i.$\Rightarrow$ ii, case $n = 3$) Choose a basis so $B =$ Diag (d, e, f), and let $A =$ Diag (a_1, a_2, a_3) be diagonal, Then consider the following equation in unknowns $\alpha_1, \alpha_2, \alpha_3$

$$\begin{aligned} \alpha_1 + d\alpha_2 + d^2\alpha_3 &= a_1 \\ \alpha_1 + e\alpha_2 + e^2\alpha_3 &= a_2 \\ \alpha_1 + f\alpha_2 + f^2\alpha_3 &= a_3 \end{aligned}$$

whose coefficient matrix

$$\begin{matrix} 1 & d & d^2 \\ 1 & e & e^2 \\ 1 & f & f^2 \end{matrix}$$

is Van der Monde, so nonsingular when d, e, f are pairwise distinct. So

$A = \alpha_1 I + \alpha_2 B + \alpha_3 B^2 \in k[B],$ ring of polynomials in B.

$\square$

Suppose on the other hand that B is a Jordan $n \times n$ block; WOLOG assume B is nilpotent. Then we have ($n = 4$)

$$B = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \quad B^2 = \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \quad B^3 = \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

It is easy to see explicitly that $A \in \mathcal{C}(B)$ iff $\exists\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in$ **C** s.t.

$$A = \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 & \alpha_1 \end{matrix}$$

so

$$A = \alpha_1 I + \alpha_2 B + \alpha_3 B^2 + \alpha_4 B^4.$$

This leads to a generalization of the diagonalization lemma:

Theorem 1.1. *Suppose the Jordan form of B has a single Jordan block for each eigenvalue. Then A commutes with B iff A is a polynomial in B.*

1.1 B Has Several Jordan blocks of EV $= \lambda$

Suppose B has several Jordan blocks with the same eigenvalue λ. Let $B = \lambda I + N, A = cI + N', N, N'$ nilpotent. Then we have

$$AB - BA = N'N - NN'$$

So WOLOG we may assume $\lambda = 0$ in studying the centralizer $\mathcal{C}_B$ of B.
Set $\mathcal{N} = \{M \in Mat_n(\mathbf{C}) \mid N^n = 0\}$, and $\mathcal{N}_B = \mathcal{N} \cap \mathcal{C}_B$.

Theorem 1.2. *Let B be nilpotent. Then $\mathcal{N}_B$ is irreducible.*

Question 1.3. Let B be a nilpotent Jordan block matrix corresponding to the partition P of n.

a. What is the set $\mathcal{Q}(P)$ of Jordan block partitions of elements of $\mathcal{N}_B$?

b. What is $Q(P)$ the Jordan block partition for the generic element of $\mathcal{N}_B$?

Asked independently by T. Košir - P. Oblak, D. I. Panyushev, and R. Basili- I-.

1.2 Tool: Powers of $A \in \mathcal{N}$, A regular.

Let $A = J_k$ $k \times k$ Jordan block EV $= 0$.
$\Rightarrow rank\, A^s = \max\{k - s, 0\}$.

Lemma 1.4. *Let* $A \in \mathcal{N} \mid UAU^{-1} = J_Q, Q = (q_1, \ldots, q_t)$ *Jordan partition* P_A.
$\Rightarrow rank\, A^{s-1} - rank\, A^s = \#\{q_i \mid s \leq q_i\}$

Cor. Corank $A = \#$ parts of Q_A.

Lemma 1.5. *Let* $n = bk + r, 0 \leq r < k$. *Let* $A \sim J_n$, *Then* P_{A^k} *has* r *parts* $b + 1$, *and* $k - r$ *parts* b .

Proof. rk $A^k = n - k$ implies P_{A^k} partitions n and has k parts. Then $(A^k)^{b+1} = 0$ and

$$rk(A^k)^b = rk(J_n)^{bk} = rk(J_n)^{n-r} = r,$$

imply P_{A^k} has r parts equal $b + 1$. Then
$rk(A^k)^{b-1} = rk(J_n)^{bk-k} = rk(J_n)^{n-r-k} = r + k$

implies there are $k - r$ parts b. $\square$

We call such P *almost rectangular* (AR): the parts differ pairwise by at most one.

Question. For which P is $Q(P) = (n)$?

Lemma 1.6. *$Q(P) = (n) \Leftrightarrow P$ is AR.*

Proof. $\Leftarrow$: If P has r parts $b+1$ and $k-r$ parts b, then B is similar to $(J_n)^k$.

$\Rightarrow$: B is similar to a matrix B' that commutes with J_n; by Theorem 1.1 $B' = u(J_n)^k$ for some k and unit u, so $P = P_B = P_{B'} = P_{{J_n}^k}$. $\square$

Ex. $P = (3,1)$ has $Q(P) = (3,1)$. $P' = (3,2,2)$ has $Q(P') = (7)$.

Note. For $P = (4), Q(P) = (4)$, but $(3,1) \notin \mathcal{Q}(P)$. Paradox: $\mathcal{N}_B$ is closed, but the set of orbits in $\mathcal{N}_B$ is not "$\overline{Q(P)}$".

Def. Set r_P = minimum number of AR subpartitions needed to write P,

$P = P_1 \cup \cdots \cup P_r$.

Ex. $P = (5, 4, 3), r_P = 2; P = (7, 6, 5, 3, 3), r_P = 3$.

Theorem 1.7. *(R. Basili). The number of parts of $Q(P)$ is r_P.*

Def. Given P' AR, $P' \subset P$, let

$s(P, P') = \#\{\text{parts of } P > \text{than any part of } P'\}$.

For $P' \subset P$ AR, the *Oblak path length* is

$$\mathrm{Ob}(P, P') = |P'| + 2s(P, P'). \qquad (1.1)$$

Theorem 1.8. *(P. Oblak) The index of $Q(P)$ (largest part) is* $\max_{P' \subset P} \mathrm{Ob}(P, P')$.

To explain this we need the poset $\mathcal{D}_P$. Let S_P be the set of integers occuring as parts of P, and write $P = \{i^{n_i}, i \in S_P\}$ (so P has n_i parts $= i$). The vertices of $\mathcal{D}_P$ correspond to the points of the Ferrer's graph of P, labelled

$$(i, u, k), i \in S_P, 1 \le u \le i, 1 \le k \le n_i.$$

Let $\nu(i, u, k) = u - (i + 1)/2$. We visualize (i, u, k) of $\mathcal{D}_P$ at $(x, y, z) = (\nu(i, u, k), i, k)$.

$$
\begin{tikzcd}
(3,1,1) \arrow[r, "w"] \arrow[d, "\beta_{3,1}"'] & (3,2,1) \arrow[r, "w"] & (3,3,1) \\
{} & & {} \arrow[u, "\alpha_{1,3}"'] \\
& (1,1,2) & \\
& (1,1,1) \arrow[u, "id"] &
\end{tikzcd}
$$

Poset $\mathcal{D}_P$ and maps for $P = (3,1,1)$

1.3 Maximal nilpotent subalgebra $\mathcal{U}_B \subset \mathcal{N}_B$.

Recall there is a canonical quotient

$$\pi : \mathcal{C}_B \to \mathcal{M}_B,$$

with kernel Jacobson radical $\mathcal{J}_B$, and image $\mathcal{M}_B$ semisimple. We choose a maximal nilpotent subalgebra $\mathcal{T}_B \subset \mathcal{M}_B$:
$\mathcal{T}_B = \{\Pi \text{ of strictly upper triang. matrices}\}$,
and set $\mathcal{U}_B = \pi^{-1}(\mathcal{T}_B)$.

Lemma 1.9. *Let $A \in \mathcal{N}_B$. Then*
$\exists C \in \mathcal{C}_B \mid CAC^{-1} \in \mathcal{U}_B$.

Let $V_{i,k} \cong k[B]/B^i$, with basis $\{v_{i,u,k} = B^{u-1}v_{i,1,k}, 1 \leq u \leq i\}$, for $i \in S_P, 1 \leq k \leq n_i$. Let $V = \oplus V_{i,k}$ and let $e_{i,k} \in \mathrm{End}(V)$ be the idempotent in $\mathrm{End}(V)$ corresponding to $V_{i,k}$. Let $E = \langle\{e_{i,k}\}\rangle$. Let $\mathcal{E}_B = E \oplus \mathcal{U}_B \subset \mathcal{C}_B$.

For $i > i'$ let $\beta_{i,i'}$ be the canonical B-module surjection $V_{i,k} \to V_{i',k}$, satisfying $\beta_{i,i'}(v_{i,1,k}) = v_{i',1,k}$.

Let $\alpha_{i',i}$ be the canonical B-module inclusion satisfying $\alpha_{i',i}(v_{i',1,k}) = v_{i,1+i-i',k}$.

Def. i is *isolated* if both $i+1, i-1 \notin S_P$. For $i \in S_P$ set i^- the next smaller, and i^+ the next larger element of S_P if they exist.

Ex. $S_P = (4,2,1)$, 4 is isolated, $4^- = 2$.

Let $\mathfrak{D}_P$ be the quiver associated to $\mathcal{E}_B$; it has one point for each part of P. The edges of $\mathfrak{D}_P$ are given by the following theorem, which also determines the edges of the poset $\mathcal{D}_P$, which has n points $\{(i,u,k)\}$.

Theorem 1.10. *$\langle e_{i,k}\ \mathcal{U}_B/\mathcal{U}_{\mathcal{B}}{}^2\ e_{i',k'}\rangle$ is one or zero-dimensional. When non-zero it has as basis the class in $\mathcal{U}_B/\mathcal{U}_{\mathcal{B}}{}^2$ of the following homomorphism in $\mathcal{U}_B$:*

i. When $i' = i^-$, the homomorphism $\beta_{i,i'}$ from $V_{i,1} \to V_{i',n_{i'}}$.

ii. When $i' = i^+$ the homomorphism $\alpha_{i,i'}$ from $V_{i,1} \to V_{i',n_{i'}}$.

iii. When $i' = i$, and $n_i > 1$, the identity homomorphism from $V_{i,k} \to V_{i,k+1}, k = 1, \ldots n_i - 1$.

iv. When $i' = i$, and i is isolated, the homomorphism J_i (Jordan nilpotent block) from $V_{i,n_i} \to V_{i,1}$.

Tbe representation $\mathcal{M}\mathfrak{D}_P$ of the quiver $\mathfrak{D}_P$ is given by the $\{V_{i,k}\}$, and the above maps together with the idempotents $\{e_{i,k}\}$.

The above maps on $\{v_{i,u,k}\}$ determine the edges of the poset $\mathcal{D}_P$ and $\mathcal{E}_B = K\mathcal{D}_P/\mathcal{I}$. That is, the maps from two paths from (i, u, k) to (i', u', k') are the same; and all compositions that go out of the poset are zero. Adjoining veriables x_p to k for each path in $\mathcal{D}_P$ a generic $A \in \mathcal{U}_B$ over $k(x)$, will be

$$A = \sum_p x_p c_p$$

where c_p is the corresponding product of maps $\beta's, \alpha's, e_{i,u,k}$ from $\mathcal{M}\mathfrak{D}_P$.

Lemma 1.11. *The index of $Q(P)$ is the length of the longest path in the poset $\mathcal{D}_P$.*

Let p_1 denote the largest part of P. We denote by v_0 the source vertex or vector $v_0 = (p_t, 0, 1)$ of $\mathcal{D}_P$, and by $\tau(v_0) = (p_t, p_t, n_{p_t})$ the sink vertex.

Lemma 1.12. *i. There is an involution τ on $\mathcal{D}_P$ that extends to $\mathcal{E}_P$:*
$\tau(i, u, k) = (i + 1 - u, u, n_i + 1 - k)$,
$\tau(\beta_{i,i'}) = \alpha_{i',i}$,

ii. The statistic $\nu(i, u, k) = u - (i+1)/2$. is nondecreasing on chains of $\mathcal{D}_P$.

iii. Every path from v_0 to $\tau(v_0)$ in $\mathcal{D}_P$ may be replaced by a τ-symmetric path from v_0 to $\tau(v_0)$ that is at least as long.

1.4 Maximal chains in $\mathcal{D}_P$

Def.. An Oblak path of $\mathcal{D}_P$ is a saturated, symmetric chain comprised of

a. Saturated path through AR $P' \subset P$.

b. Two "tails" T and $\iota(T)$. T is through all initial vertices $(i, 1, k)$ above P'. $\iota(T)$ is the path back through all terminal vertices (i, i, k) above P".

The following result was first shown by P. Oblak.

Theorem 1.13. *There is an Oblak chain between v_0 and $\tau(v_0)$ in $\mathcal{D}_P$ that has maximum length among all chains from v_0 to $\tau(v_0)$.*

Proof. We give a new proof using Lemma 1.12, and an induction. □

This proves Oblak's theorem about the index of $Q(P)$.

1.5 When is the ring $k[A, B]$ Gorenstein?

We say that P is *stable* if $Q(P) = P$.

Lemma 1.14. *(R. Basili, A.I) P is stable iff its parts differ pairwise by at least two.*

We denote by $H(\mathcal{A})$ the Hilbert function of an algebra $\mathcal{A}$. We denote by $P(H)$ the partition dual to $(1, H(1), H(2), ..., H(n))$.

Example 1.15. For $H = (1, 2, 3, 3, 2, 2, 1)$, $P(H) = (7, 5, 2)$. For $H = (1, 2, 3, 2, 1)$, $P(H) = (5, 3, 1)$. For $H = (1, 2, 3, 1)$, $P(H) = (4, 2, 1)$ (is not stable).

Theorem 1.16. *(R. Basili, AI). (*char $k = 0$ *or* char $k > n$*). Let* $\dim k[A, B] = n$*. Then for an open dense set of $\lambda \in k$, $A+\lambda B$ has Jordan block partition $P(H)$, $H = H(k[A, B])$.*

Theorem 1.17. *(P. Oblak, T. Košir). For A generic, $k[A, B]$ is Gorenstein.*

Corollary 1.18. *$Q(P)$ is stable.*

This follows from Theorems 1.16,1.17, and

Theorem 1.19. *(F.H.S. Macaulay). Let $\mathcal{A}$ be a CI Artinian quotient of $k\{x, y\}$, local ring. Then*

$H(\mathcal{A}) = (1, 2, 3, \ldots, d, t_d, \ldots, t_j)$ *where* $\forall i \ \mid t_i - t_{i+1} \mid \leq 1$.

- $P(H)$ has parts differing by at least two!

Note. Let $i \in S_P$. The i-th semisimple part of $C \in \mathcal{C}_B$ is the restriction of C to the span $\langle v_{i,1,1}, \dots v_{i,1,n_i} \rangle$. Thus $A \in \mathcal{U}_B \equiv A$ is upper triangular on each span. Equivalently, A has no downward components on the span.

We now give a simple criterion.

Theorem 1.20. *Assume that $A \in \mathcal{U}_B$ has nonzero components on each $\beta_{i,j}, \alpha_{i,j}$ and each map $t_{i,u} : v_{i,1,u} \rightarrow v_{i,1,u+1}$. Then $k[A, B]$ has v_0 as cyclic vector, and $\langle \tau(v_0) \rangle$ as socle, so is Gorenstein.*

Proof. $(0 : B)$ is just the right side of $\mathcal{D}_B$ $(\{v_{i,i,k}\})$. On the right side A, having only nonzero components on $\alpha' s$ and on $\tau(t_{i,u})$ satisfies

$(0 : A) \cap (0 : B) = \tau(v_0)$. $\square$

1.6 Connection with Hilbert scheme

The natural connection between commuting $n \times n$ nilpotent matrices and the fibre of the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ over a point p of $\mathbb{A}^2$ was noted by H. Nakajima; and used by V. Baranovsky, R. Basili, and A. Premet, to study the irreducibility of the variety of pairs of commuting nilpotent matrices, using J. Briançon's work, or vice versa. Since a pair of commuting matrices may not have a cyclic vector, the theory of pairs and triples of commuting nilpotent matrices is related to that of Hilbert schemes, but is not "isomorphic".