Computing Noncommutative Massey Products

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Deformation problem

We fix a field k.

Our example:

- $A = k[x, y]$ is the commutative coordinate ring of the affine plane
- $M = A/(x^2, y)$ is considered as a right A-module

We want to compute:

1. The pro-representing hull $H^c(M)$ of the commutative deformation functor Def^c_M of the right A-module M
2. The pro-representing hull $H(M)$ of the noncommutative deformation functor Def_M of the right A-module M
3. The commutative versal family $\mathcal{M}^c$ defined over $H^c(M)$
4. The noncommutative versal family $\mathcal{M}$ defined over $H(M)$

The tangent space and obstruction space

We fix a free resolution $(L_\bullet, d_\bullet)$ of the right A-module M:

$$0 \leftarrow M \leftarrow A \xleftarrow{(\,x^2\;\; y\,)\cdot} A^2 \xleftarrow{\begin{pmatrix} y \\ -x^2 \end{pmatrix}\cdot} A \leftarrow 0$$

To compute $\mathrm{Ext}^n_A(M, M)$, we consider the complex $\mathrm{Hom}_A(L_\bullet, M)$:

$$M \xrightarrow{\cdot(\,x^2\;\; y\,)} M^2 \xrightarrow{\cdot\begin{pmatrix} y \\ -x^2 \end{pmatrix}} M \to 0$$

The maps in this complex are zero, and the tangent space H^1 and the obstruction space H^2 for either of the deformation functors are given by

$$\boxed{H^1 = \mathrm{Ext}^1_A(M, M) = M^2, \quad H^2 = \mathrm{Ext}^2_A(M, M) = M}$$

where $\dim_k H^1 = 4$ and $\dim_k H^2 = 2$ since $M \simeq k \oplus kx$ has dimension two.

The pro-representing hulls

From the dimensions of the tangent space and obstruction space, we may conclude that the hulls have the following form:

Pro-representing hulls

There are commutative power series f_1^c, f_2^c and noncommutative power series f_1, f_2 such that

$$H^c(M) = k[[t_1, t_2, t_3, t_4]]/(f_1^c, f_2^c)$$
$$H(M) = k\langle\langle t_1, t_2, t_3, t_4\rangle\rangle/(f_1, f_2)$$

We must use commutative and noncommutative Massey products to compute these power series, and the versal families will be discovered as a side product of these computations.

The Yoneda DGA

We know that $H^n(Y^\bullet) \simeq \mathrm{Ext}^n_A(M,M)$, where $Y^\bullet = \mathrm{Hom}^{(\bullet)}_A(L_\bullet, L_\bullet)$ is the Yoneda DGA (differential graded algebra). The Yoneda DGA is given by

$$Y^n = \mathrm{Hom}^{(n)}_A(L_\bullet, L_\bullet) = \coprod_{i \geq 0} \mathrm{Hom}_A(L_{n+i}, L_i)$$

for $n \geq 0$, and for any element $\phi = (\phi_i)_{i \geq 0} \in Y^n$ with $\phi_i : L_{i+n} \to L_i$, the differential $d_n : Y^n \to Y^{n+1}$ is given by

$$d^n(\phi) = \psi = (\psi_i)_{i \geq 0}, \text{ with } \psi_i = \phi_i d_{n+i} + (-1)^{n+1} d_i \phi_{i+1}$$

Representations of cohomology classes when $L_i = 0$ for $i > 2$

An element in $H^1 = H^1(Y^\bullet)$ can be represented by a pair $(\phi_0, \phi_1) \in Y^1$ such that $d_0\phi_1 + \phi_0 d_1 = 0$. An element in $H^2 = H^2(Y^\bullet)$ can be represented by an element $\omega_0 \in Y^2$. The multiplication $Y^1 \otimes_k Y^1 \to Y^2$ is given by

$$(\phi_0, \phi_1) \cdot (\psi_0, \psi_1) = \phi_0 \circ \psi_1$$

Yoneda representations of tangent vectors

Let us choose a k-base of the tangent space $H^1 = \mathrm{Ext}^1_A(M, M) = M^2$ consisting of

$$t_1^* = (1, 0), \quad t_2^* = (x, 0), \quad t_3^* = (0, 1), \quad t_4^* = (0, x)$$

and a cocycle $\alpha(i) = \begin{pmatrix} \alpha(i)_0 & \alpha(i)_1 \end{pmatrix} \in Y^1$ that lifts the cohomology class $t_i^* \in H^1(Y^\bullet)$ for $i = 1, 2, 3, 4$:

Yoneda representatives of tangent vectors

$$\alpha(1) = \{ \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \} \qquad \alpha(3) = \{ \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$$

$$\alpha(2) = \{ \begin{pmatrix} x & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x \end{pmatrix} \} \qquad \alpha(4) = \{ \begin{pmatrix} 0 & x \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} \}$$

Yoneda representations of obstruction vectors

Let us choose a k-base of the obstruction space $H^2 = \operatorname{Ext}^2_A(M,M) = M$ consisting of

$$s_1^* = (1), \quad s_2^* = (x)$$

and a cocyle $\omega(i) = \big(\omega(i)_0\big) \in Y^2$ that lifts the cohomology class $s_i^* \in H^2(Y^\bullet)$ for $i = 1,2$:

Yoneda representatives of obstruction vectors

$$\omega(1) = \{\big(1\big)\} \qquad \omega(2) = \{\big(x\big)\}$$

Versal family at the tangent level

At the tangent level, $H_2^c = H^c/I(H^c)^2$ and $H_2 = H/I(H)^2$ both equal $k[\epsilon] = k[\epsilon_1, \dots, \epsilon_4]$, where $\epsilon_i = \overline{t_i}$ and $\epsilon_i \epsilon_j = 0$ for all i, j.

Lifting of families

We write $A[\epsilon] = k[\epsilon_1, \dots, \epsilon_4] \otimes_k A = H_2 \otimes_k A = H_2^c \otimes_k A$, and define liftings of the A-linear differentials d_0 and d_1 to $A[\epsilon]$ by

$$d_0[\epsilon] = d_0 + \sum_{1 \le m \le 4} \epsilon_m \, \alpha(m)_0 = \begin{pmatrix} x^2 + \epsilon_1 + \epsilon_2 x & y + \epsilon_3 + \epsilon_4 x \end{pmatrix}$$

$$d_1[\epsilon] = d_1 + \sum_{1 \le m \le 4} \epsilon_m \, \alpha(m)_1 = \begin{pmatrix} y + \epsilon_3 + \epsilon_4 x \\ -x^2 - \epsilon_1 - \epsilon_2 x \end{pmatrix}$$

Versal family at the tangent level

We consider the following sequence of maps, where $M[\epsilon] = \text{coker}(d_0[\epsilon])$:

$$0 \leftarrow M[\epsilon] \leftarrow A[\epsilon] \xleftarrow{d_0[\epsilon]} A[\epsilon]^2 \xleftarrow{d_1[\epsilon]} A[\epsilon] \leftarrow 0 \qquad (1)$$

By construction, this is a complex. In fact, it is instructional to compute the matrix product

$$d_0[\epsilon]d_1[\epsilon] = (\epsilon_1\epsilon_3 - \epsilon_3\epsilon_1) + (\epsilon_1\epsilon_4 - \epsilon_4\epsilon_1 + \epsilon_2\epsilon_3 - \epsilon_3\epsilon_2)x + (\epsilon_2\epsilon_4 - \epsilon_4\epsilon_2)x^2$$

It is zero since $\epsilon_i\epsilon_j = 0$ in $k[\epsilon]$.

Conclusion: Versal families at the tangent level

- The complex (1) is a free resolution of $M[\epsilon]$ that lifts $(L_\bullet, d_\bullet)$ to $A[\epsilon]$
- The versal families at the tangent level are $\mathcal{M}_2 = \mathcal{M}_2^c = M[\epsilon]$

Cup products

The Massey products $\langle \alpha(i), \alpha(j) \rangle$ of order two are called cup products.

Definition of cup products

The commutative and noncommutative cup products are defined in terms of the multiplication in the Yoneda DGA:

$$\langle \alpha(i), \alpha(j) \rangle^c = \alpha(i)\,\alpha(j) + \alpha(j)\,\alpha(i)$$
$$\langle \alpha(i), \alpha(j) \rangle = \alpha(i)\,\alpha(j)$$

The cup products give second order approximations $f_i = f_i^2 + I(H)^3$ and $f_i^c = (f_i^c)^2 + I(H^c)^3$ of the power series, where

$$f_i^2 = \sum_{1 \le m,n \le 4} \omega(i)^*(\langle \alpha(m), \alpha(n) \rangle)\, t_m t_n$$
$$(f_i^c)^2 = \sum_{1 \le m \le n \le 4} \omega(i)^*(\langle \alpha(m), \alpha(n) \rangle^c)\, t_m t_n$$

Computation of cup products

We compute the noncommutative cup products that are non-zero in Y^2:

$$\begin{aligned}
\langle\alpha(1),\alpha(3)\rangle &= (1) = \omega(1) & \langle\alpha(3),\alpha(1)\rangle &= (-1) = -\omega(1)\\
\langle\alpha(1),\alpha(4)\rangle &= (x) = \omega(2) & \langle\alpha(4),\alpha(1)\rangle &= (-x) = -\omega(2)\\
\langle\alpha(2),\alpha(3)\rangle &= (x) = \omega(2) & \langle\alpha(3),\alpha(2)\rangle &= (-x) = -\omega(2)\\
\langle\alpha(2),\alpha(4)\rangle &= (x^2) & \langle\alpha(4),\alpha(2)\rangle &= (-x^2)
\end{aligned}$$

We notice that all commutative cup products are zero in Y^2.

Second order approximations

$$\begin{aligned}
f_1^2 &= t_1t_3 - t_3t_1 & (f_1^c)^2 &= 0\\
f_2^2 &= t_1t_4 - t_4t_1 + t_2t_3 - t_3t_2 & (f_2^c)^2 &= 0
\end{aligned}$$

Cup products and lifting of complexes

Let us try to lift the complex (1) from the tangent level to $k[[t_1, \ldots, t_4]]$ or $k\langle\langle t_1, t_2, t_3, t_4\rangle\rangle$ by replacing $d_i[\epsilon]$ with $d_i^1(t)$:

$$d_0^1(t) = d_0 + \sum_{1 \le m \le 4} t_m \, \alpha(m)_0 = \begin{pmatrix} x^2 + t_1 + t_2 x & y + t_3 + t_4 x \end{pmatrix}$$

$$d_1^1(t) = d_1 + \sum_{1 \le m \le 4} t_m \, \alpha(m)_1 = \begin{pmatrix} y + t_3 + t_4 x \\ -x^2 - t_1 - t_2 x \end{pmatrix}$$

The obstruction for this to be a lifting of complexes is

$$d_0^1(t) d_1^1(t) = (t_1 t_3 - t_3 t_1) + (t_1 t_4 - t_4 t_1 + t_2 t_3 - t_3 t_2) x + (t_2 t_4 - t_4 t_2) x^2$$

Note that the coefficients in front of $1 = \omega(1)$ and $x = \omega(2)$ are the second order approximations, and that the obstruction vanishes in $k[[t_1, \ldots, t_4]]$.

The commutative case

In the commutative situation, we have $d_0^1(t)d_1^1(t) = 0$. We may consider $d_i(t) = d_i^1(t)$ as a matrix with coefficients in $A[[t]] = k[[t_1, \ldots, t_4]]\widehat{\otimes}_k A$. Moreover, we consider the complex

$$0 \leftarrow M(t) \leftarrow A[[t]] \xleftarrow{d_0(t)} A[[t]]^2 \xleftarrow{d_1(t)} A[[t]] \leftarrow 0 \qquad (2)$$

where $M(t) = \operatorname{coker}(d_0(t))$.

Commutative versal family

- The complex (2) is a free resolution of $M(t)$ that lifts (1) to $A[[t]]$
- The pro-representing hull of the commutative deformation functor Def_M^c is $H(M)^c = k[[t_1, \ldots, t_4]]$, and $f_1^c = f_2^c = 0$
- The versal family in the commutative situation is $\mathcal{M}^c = M(t)$

The noncommutative case

In the noncommutative situation, the obstruction for lifting the complex (1) to $T^1 = k\langle\langle t_1, t_2, t_3, t_4\rangle\rangle$ does not vanish, since

$$\begin{aligned} d_0^1(t)d_1^1(t) &= (t_1t_3 - t_3t_1) + (t_1t_4 - t_4t_1 + t_2t_3 - t_3t_2)x + (t_2t_4 - t_4t_2)x^2 \\ &= f_1^2 \cdot \omega(1) + f_2^2 \cdot \omega(2) + (t_2t_4 - t_4t_2)x^2 \end{aligned}$$

where $\left(x^2\right)$ is a coboundary in Y^2. It follows that at the next level, where $I(H)^3 = 0$, we must kill the obstructions by forcing $f_1^2 = f_2^2 = 0$:

$$\boxed{\begin{aligned} H_3 &= T^1/(I(T^1)^3 + (f_1^2, f_2^2)) \\ &= k\langle t_1, t_2, t_3, t_4\rangle/((t_1, t_2, t_3, t_4)^3, [t_1, t_3], [t_1, t_4] + [t_2, t_3]) \end{aligned}}$$

where we write $[t_i, t_j] = t_it_j - t_jt_i$ for all i, j.

Idea: Immediately defined third order Massey products

A third order Massey product $\langle \alpha(i), \alpha(j), \alpha(k) \rangle$ is immediately defined if the intermediary cup products are coboundaries:

$$\langle \alpha(i), \alpha(j) \rangle = d(\alpha(i,j))$$
$$\langle \alpha(j), \alpha(k) \rangle = d(\alpha(j,k))$$

In that case, $\{\alpha(i), \alpha(j), \alpha(k), \alpha(i,j), \alpha(j,k)\}$ is a defining system for the third order Massey product. Given a defining system, the third order Massey product is defined by

$$\langle \alpha(i), \alpha(j), \alpha(k) \rangle = \alpha(i)\alpha(j,k) + \alpha(i,j)\alpha(k)$$

Even if the third order Massey product is defined, it may depend on the defining system.

Defining systems for third order Massey products

Let $\overline{B}_1$ be the set of monomials in the variables $t_1, \dots, t_4$ of order at most one, and extend this to a monomial k-base $\overline{B}_2 = \overline{B}_1 \cup B_2$ for H_3, with

$$B_2 = \{t_i t_j : 1 \leq i, j \leq 4\} \setminus \{t_1 t_3, t_1 t_4\}$$

We use the quadratic relations $t_1 t_3 = t_3 t_1$ and $t_1 t_4 = t_4 t_1 - t_2 t_3 + t_3 t_2$ to express any quadratic monomial $\underline{t} = t_i t_j$ as

$$\underline{t} = \sum_{\underline{t}' \in B_2} \beta(\underline{t}, \underline{t}')\, \underline{t}'$$

with $\beta(\underline{t}, \underline{t}') \in k$. For any $t_m t_n \in B_2$, there exists a (non-unique) element $\alpha(m, n) \in Y^1$ such that

$$\sum_{1 \leq i,j \leq 4} \langle \alpha(i), \alpha(j) \rangle\, \beta(t_i t_j, t_m t_n) = -d(\alpha(m, n))$$

Computing defining systems

For $\underline{t} = t_m t_n \in B_2$, we compute the right-hand side of the equation

$$d(\alpha(m,n)) = -\sum_{1 \le i,j \le 4} \langle \alpha(i), \alpha(j) \rangle \, \beta(t_i t_j, t_m t_n)$$

Using the cup-products computed earlier, we see that the right-hand side vanishes in all cases except these:

$$d(\alpha(2,4)) = -\langle \alpha(2), \alpha(4) \rangle = (-x^2)$$
$$d(\alpha(4,2)) = -\langle \alpha(4), \alpha(2) \rangle = (x^2)$$

We choose $\alpha(m,n) = 0$ for all $t_m t_n \in B_2$ with $(m,n) \neq (2,4), (4,2)$, and

$$\alpha(2,4) = \{ \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}, \quad \alpha(4,2) = \{ \begin{pmatrix} 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

Then $D(3) = \{\alpha(m,n) : t_m t_n \in B_2\} \cup \{\alpha(m) : t_m \in \overline{B}_1\}$ is a defining system for third order Massey products.

Third order Massey products

The third order Massey product $\langle \alpha(p), \alpha(q), \alpha(r) \rangle$ is defined for any monomial $\underline{t} = t_p t_q t_r$ in B_3', where

$$B_3' = \{t_i t_j t_k : 1 \le i, j, k \le 4\} \setminus \{t_i t_1 t_3, t_i t_1 t_4, t_1 t_3 t_i, t_1 t_4 t_i : 1 \le i \le 4\}$$

Let $\overline{B_3'} = B_3' \cup \overline{B}_2$. For any monomial $\underline{t}$ of degree at most three, we have

$$\underline{t} = \sum_{\underline{t}' \in \overline{B_3'}} \beta'(\underline{t}, \underline{t}')\underline{t}' + \sum_{1 \le i \le 2} \beta'(\underline{t}, i) f_i^2$$

with $\beta(\underline{t}, \underline{t}'), \beta'(\underline{t}, i) \in k$. The third order Massey products are given by

$$\boxed{\langle \alpha(p), \alpha(q), \alpha(r) \rangle = \sum_{\substack{\underline{t} = \underline{t}' \underline{t}'' \\ \underline{t}', \underline{t}'' \in \overline{B}_2}} \alpha(\underline{t}')\alpha(\underline{t}'')\, \beta'(\underline{t}, t_p t_q t_r)}$$

Computing third order Massey products

In the Yoneda DGA, the only non-zero products $\alpha(\underline{t}, \underline{t}')$ with $\underline{t}, \underline{t}' \in \overline{B}_2$ are

$$\alpha(2,4)\alpha(1) = (-1) = -\omega(1) \qquad \alpha(4,2)\alpha(1) = (1) = \omega(1)$$
$$\alpha(2,4)\alpha(2) = (-x) = -\omega(2) \qquad \alpha(4,2)\alpha(2) = (x) = \omega(2)$$

Since the monomials $t_2 t_4 t_1, t_2 t_4 t_2, t_4 t_2 t_1, t_4 t_2 t_2$ are not involved in any of the relations, the only non-zero third order Massey products are

$$\langle \alpha(2), \alpha(4), \alpha(1) \rangle = -\omega(1) \qquad \langle \alpha(4), \alpha(2), \alpha(1) \rangle = \omega(1)$$
$$\langle \alpha(2), \alpha(4), \alpha(2) \rangle = -\omega(2) \qquad \langle \alpha(4), \alpha(2), \alpha(2) \rangle = \omega(2)$$

Third order approximations

$$f_1^3 = [t_1, t_3] - [t_2, t_4] t_1$$
$$f_2^3 = [t_1, t_4] + [t_2, t_3] - [t_2, t_4] t_2$$

Versal family at the third level

Using the defining system $D(3)$, we find a lifting of the versal family $\mathcal{M}_2$ at the tangent level to a versal family $\mathcal{M}_3$ defined over H_3:

Lifting of families to H_3

We define matrices with coefficients in $k\langle\langle t_1, \dots, t_4 \rangle\rangle \widehat{\otimes}_k A$:

$$
\begin{aligned}
d_0^2 &= d_0^1 + \sum_{t_m t_n \in B_2} t_m t_n \, \alpha(m,n)_0 \\
&= \begin{pmatrix} x^2 + t_1 + t_2 x & y + t_3 + t_4 x + t_2 t_4 - t_4 t_2 \end{pmatrix} \\
d_1^2 &= d_1^1 + \sum_{t_m t_n \in B_2} t_m t_n \, \alpha(m,n)_1 = \begin{pmatrix} y + t_3 + t_4 x \\ -x^2 - t_1 - t_2 x \end{pmatrix}
\end{aligned}
$$

Considered as matrices with coefficients in $H_3 \otimes_k A$, this is a lifting of the complex (1) at the tangent level to $H_3 \otimes_k A$, and $\mathcal{M}_3 = \operatorname{coker}(d_0^2)$.

Third order Massey products and lifting of complexes

Let us compute the matrix product $d_0^2\, d_1^2$ in $k\langle\langle t_1, \ldots, t_4\rangle\rangle \widehat{\otimes}_k A$, the obstruction for lifting the complex to $k\langle\langle t_1, \ldots, t_4\rangle\rangle$:

$$\begin{aligned} d_0^2 d_1^2 &= d_0^1 d_1^1 + [t_2, t_4](-x^2 - t_1 - t_2 x) \\ &= ([t_1, t_3] - [t_2, t_4]t_1) + ([t_1, t_4] + [t_2, t_3] - [t_2, t_4]t_2)\, x \\ &= f_1^3 \omega(1) + f_2^3 \omega(2) \end{aligned}$$

We must kill the obstructions by forcing $f_1^3 = f_2^3 = 0$, and then we are done:

$$\boxed{\begin{aligned} H &= T^1/(f_1^3, f_2^3)) \\ &= k\langle\langle t_1, t_2, t_3, t_4\rangle\rangle/([t_1, t_3] - [t_2, t_4]t_1, [t_1, t_4] + [t_2, t_3] - [t_2, t_4]t_2) \end{aligned}}$$

With $d_0 = d_0^2$ and $d_1 = d_1^2$, we then have $d_0 d_1 = 0$ in $H \widehat{\otimes}_k A$.

Conclusions in the noncommutative case

Using noncommutative Massey products up to order three, we have found:

Results in the noncommutative case

- The relations are given by

$$f_1 = f_1^3 = [t_1, t_3] - [t_2, t_4]t_1, \quad f_2 = f_2^3 = [t_1, t_4] + [t_2, t_3] - [t_2, t_4]t_2$$

- The pro-representing hull is given by

$$H = k\langle\langle t_1, t_2, t_3, t_4\rangle\rangle/([t_1, t_3] - [t_2, t_4]t_1, [t_1, t_4] + [t_2, t_3] - [t_2, t_4]t_2)$$

- The versal family is given by $\mathcal{M} = \text{coker}(d_0)$, with free resolution

$$0 \leftarrow \mathcal{M} \leftarrow H\widehat{\otimes}_k A \xleftarrow{d_0} (H\widehat{\otimes}_k A)^2 \xleftarrow{d_1} H\widehat{\otimes}_k A \leftarrow 0$$

References

Noncommutative deformations of modules

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