Generaliserte Massey-produkter. Definisjon og eksempler.

$(A^\cdot, d_\cdot)$ differensialgradert k-algebra. $\bar\alpha_{e_1}, \bar\alpha_{e_2}, \dots, \bar\alpha_{e_d} = \underline{\alpha} \in H^1(A^\cdot)$.

$\langle \underline{\alpha}; \underline{n}\rangle$, $\underline{n} \in B_N' \subseteq \{\underline{n} \in \mathbb{N}^d \mid |n| \leq N\}$?

Definisjon. $S_2 = k[[u_1, \dots, u_d]]/\underline{m}^2 = k[[\underline{u}]]/\underline{m}^2 \leftarrow k[[\underline{u}]]/\underline{m}^3 = R^3$.

$\bar B_1 = \{\underline{n} \in \mathbb{N}^d \mid |n| \leq 1\}$, $B_2' = \{\underline{n} \in \mathbb{N}^d \mid |n| = 2\}$, $\bar B_2' = \bar B_1 \cup B_2'$

$$\langle \underline{\alpha}; \underline{n}\rangle = \overline{y(n)}, \quad y(\underline{n}) = \sum_{\substack{\underline{m}_1 + \underline{m}_2 = \underline{n}\\ |\underline{m}_i| = 1}} \alpha_{\underline{m}_1} \cdot \alpha_{\underline{m}_2}, \quad \underline{n} \in B_2'$$

$$\operatorname{Span}_{\underline{n} \in B_2'}(\langle \underline{\alpha}; \underline{n}\rangle) = \sum_{i=1}^{r} k y_i^*, \quad f_i^2 = \sum_{\underline{n} \in B_2'} y_i(\langle \underline{\alpha}; \underline{n}\rangle)\, \underline{u}^{\underline{n}}, \quad i = 1, \dots, r.$$

$S_3 = R_3 / (f_1^2, \dots, f_r^2)$, $B_2 \subseteq B_2'$ monomial basis for $\underline{m}^2/\underline{m}^3 + (f_1^2, \dots, f_r^2)$.

$\bar B_2 = \bar B_1 \cup B_2$. $\underline{n} \in \mathbb{N}^d$, $|n| \leq 3 \Rightarrow \exists!\ \underline{u}^{\underline{n}} = \sum_{\underline{m} \in \bar B_2} \beta_{\underline{n},\underline{m}}\, \underline{u}^{\underline{m}}$.

$$\text{I } S_3:\quad 0 = \sum_{j=1}^{r} y_j^* \otimes f_j = \sum_{j=1}^{r} y_j^* \otimes \sum_{\underline{n} \in B_2'} y_i(\langle \underline{\alpha}; \underline{n}\rangle)\, \underline{u}^{\underline{n}}$$

$$= \sum_{\underline{n} \in B_2'} \langle \underline{\alpha}; \underline{n}\rangle \otimes \underline{u}^{\underline{n}} = \sum_{\underline{n} \in B_2'} \langle \underline{\alpha}; \underline{n}\rangle \otimes \sum_{\underline{m} \in \bar B_2} \beta_{\underline{n},\underline{m}}\, \underline{u}^{\underline{m}}$$

$$= \sum_{\underline{m} \in \bar B_2} \Big(\sum_{\underline{n} \in B_2'} \beta_{\underline{n},\underline{m}} \langle \underline{\alpha}; \underline{n}\rangle\Big) \otimes \underline{u}^{\underline{m}} = \sum_{\underline{m} \in \bar B_2} b_{\underline{m}} \otimes \underline{u}^{\underline{m}} \Rightarrow b_{\underline{m}} = 0.$$

$\alpha_{\underline{m}} \in A^1$, $d(\alpha_{\underline{m}}) = -b_{\underline{m}}$. $R_4 = k[[\underline{u}]]/\underline{m}^4 + \underline{m}(f_1^2, \dots, f_r^2)$.

$\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_3'}$ monomial basis for $\underline{m}^3/\underline{m}^4 + \underline{m}^3 \cap \underline{m}(f_1^2, \dots, f_r^2)$ slik at

for $\underline{n} \in B_3'$, $\underline{u}^{\underline{n}} = u_k \cdot \underline{u}^{\underline{m}}$ for en $0 \leq k \leq d$ og en $\underline{m} \in B_2$.

Anta $R_{n+1} = k[[\underline{u}]]/\underline{m}^{n+1} + \underline{m}(f_1^{n-1}, \dots, f_r^{n-1}) \xrightarrow{\pi_{n+1}'} k[[\underline{u}]]/(\underline{m}^n + (f_1^{n-1}, \dots, f_r^{n-1}))$,

B_n, $\bar B_n$, B_{n+1}' $\{\alpha_{\underline{m}}\}_{\underline{m} \in \bar B_{n-1}}$ konstruert for $1 \leq n \leq N$.

$\ker \pi_{n+1}' = \underline{m}^n/(\underline{m}^{n+1} + \underline{m}^n \cap \underline{m}(f_1^{n-1}, \dots, f_r^{n-1})) \oplus (f_1^{n-1}, \dots, f_r^{n-1})/\underline{m}(f_1^{n-1}, \dots, f_r^{n-1})$,

$\bar{B}'_{N+1} = \bar{B}_N \cup B'_{N+1}$. $\underline{n} \in \bar{B}'_{N+1} \Rightarrow \exists!\ \underline{u}^{\underline{n}} = \sum_{\underline{m} \in \bar{B}'_{N+1}} \beta'_{\underline{m},\underline{n}} \underline{u}^{\underline{m}} + \sum_{j=1}^{r} \beta_{\underline{n},j} f_j^{N-1}$

De <u>N-te ordens Massey-produktene</u> for $\underline{n} \in B'_{N+1}$ er

$$\langle \underline{\alpha}; \underline{n} \rangle = \overline{y(n)}, \quad y(\underline{n}) = \sum_{|\underline{m}| \leq N+1} \sum_{\substack{\underline{m}_1 + \underline{m}_2 = \underline{m} \\ \underline{m}_i \in \bar{B}_N}} \beta'_{\underline{m},\underline{n}} \alpha_{\underline{m}_1} \cdot \alpha_{\underline{m}_2}$$

Sett $f_j^N = f_j^{N-1} + \sum_{\underline{n} \in B'_N} y_j(\langle \underline{\alpha}; \underline{n} \rangle) \underline{u}^{\underline{n}}$, $j = 1, \dots, r$,

$$R_{N+2} = k[[\underline{u}]] / (\underline{m}^{N+2} + \underline{m}(f_1^N, \dots, f_r^N))$$
$$\downarrow \pi'_{N+2}$$

$$S_{N+1} = R_{N+1}/(f_1^N, \dots, f_r^N) = k[[\underline{u}]]/\underline{m}^{N+1} + (f_1^N, \dots, f_r^N) \xrightarrow{\pi_{N+1}} S_N.$$

$\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_{N+1}}$ monomial basis for $\ker \pi_{N+1}$ s.a. $B_{N+1} \subseteq B'_{N+1}$,

$\bar{B}_{N+1} = \bar{B}_N \cup B_{N+1}$. $\underline{n} \in \mathbb{N}^d$, $|\underline{n}| \leq N \Rightarrow \exists!\ \underline{u}^{\underline{n}} = \sum_{\underline{m} \in \bar{B}_{N+1}} \beta_{\underline{n},\underline{m}} \underline{u}^{\underline{m}}$,

$\underline{m} \in B_{N+1} \Rightarrow b_{\underline{m}} = \sum_{l=0}^{N} \sum_{\underline{n} \in B'_{2+l}} \beta_{\underline{n},\underline{m}} \langle \underline{\alpha}; \underline{n} \rangle = 0.$

Velg $\alpha_{\underline{m}} \in A^1$, $d(\alpha_{\underline{m}}) = -b_{\underline{m}}$.

$\{\alpha_{\underline{m}}\}_{\underline{m} \in \bar{B}_{N+1}}$ kalles et <u>definerende system</u> for Massey produktene

$\langle \underline{\alpha}; \underline{n} \rangle$, $\underline{n} \in B'_{N+2}$.

$\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B'_{N+2}}$ monomial basis for $\underline{m}^{N+1}/\underline{m}^{N+2} + \underline{m}^{N+1} \cap \underline{m}(f_1^N, \dots, f_r^N)$

slik at $\underline{n} \in B'_{N+2} \Rightarrow \underline{u}^{\underline{n}} = u_k \cdot \underline{u}^{\underline{m}}$ for en k, $1 \leq k \leq d$ <u>og</u> $\underline{m} \in B_{N+1}$.

Konstruksjonen er komplett ved induksjon.

Definisjon. Konstruksjonen veldefinert $\Leftrightarrow$ $y(\underline{n})$ korand og uavhengig av valg av definerende systemer $\Leftrightarrow$ $(A^\circ, d_\cdot)$ OS-algebra (obstruction situation algebra). $\hat{H}_{\underline{\alpha}} = k[[\underline{u}]]/(f_1,\dots,f_r)$,
$f_i = \sum_{l=2}^{\infty} \sum_{\underline{n}\in B_l^i} y_i(\langle\underline{\alpha};\underline{n}\rangle)\,\underline{u}^{\underline{n}}$ relasjonsalgebraen.

Obstruksjonsteori

Proposisjon. La $\alpha_{e_1},\dots,\alpha_{e_d} \in \mathrm{Ext}^1_A(M,M) = T_{\mathrm{Def}_M} = \mathrm{Def}_M(k[\varepsilon])$, og la $M_2 \in \mathrm{Def}_M(S_2)$ være det korresponderende elementet i $\mathrm{Def}_M(S_2)(\cong \mathrm{Mor}(S_2,\hat{H}_2))$. Da er $o(M_2,\pi_2')$ gitt ved Massey-produktene i $\mathrm{Hom}^\bullet(L_\cdot,L_\cdot)$ der $0\leftarrow M\leftarrow L_\cdot$ er en fri oppløsning. Spesielt, hvis $\alpha_{e_1},\dots,e_{e_d}$ er en basis for $\mathrm{Ext}^1_A(M,M)$, så er $\hat{H}(M)$ relasjonsalgebraen. ($\mathrm{Hom}^p(L_\cdot,L_\cdot)\xrightarrow{d^p}\mathrm{Hom}^{p+1}(L_\cdot,L_\cdot)$, $d^p(\{\xi_i\}) = \{\xi\circ d - (-1)^p d\circ\xi\}$).

Eks 1: $A = k[x,y]$, $M = A/(x^2,y)$

$$0 \leftarrow M \leftarrow A \xleftarrow{(x^2\ y)} A^2 \xleftarrow{\begin{pmatrix}-y\\ x^2\end{pmatrix}} A \leftarrow 0$$

$$\Rightarrow 0 \to \mathrm{Hom}(M,M) \to M \xrightarrow[0=\begin{pmatrix}x^2\\ y\end{pmatrix}]{} M^2 \xrightarrow[(-y\ x^2)=0]{} M \to 0$$

$\Rightarrow \mathrm{Ext}^1_A(M,M) \cong M^2$, $\mathrm{Ext}^2_A(M,M)\cong M$.

Yoneda form:

$$\begin{tikzcd}
0 & M \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x^2\ y)"'] \arrow[dl, "(1\ 0)\ (x\ 0)\ (0\ 1)\ (0\ x)"] & A \arrow[l, "\begin{pmatrix}-y\\ x^2\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}0\\ 1\end{pmatrix}\begin{pmatrix}0\\ x\end{pmatrix}\begin{pmatrix}-1\\ 0\end{pmatrix}\begin{pmatrix}-x\\ 0\end{pmatrix}"] & 0 \arrow[l] \\
0 & M \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x^2\ y)"] & A \arrow[l, "\begin{pmatrix}-y\\ x^2\end{pmatrix}"] & 0 \arrow[l]
\end{tikzcd}$$

Cup-produktene

$\alpha_{e_1}\alpha_{e_3} = -1$, $\alpha_{e_1}\alpha_{e_4} = -x$, $\alpha_{e_2}\alpha_{e_3} = -x$, $\alpha_{e_2}\alpha_{e_4} = -x^2 = 0$,

$\alpha_{e_3}\alpha_{e_1} = 1$, $\alpha_{e_3}\alpha_{e_2} = x$, $\alpha_{e_4}\alpha_{e_1} = x$, $\alpha_{e_4}\alpha_{e_2} = x^2 = 0$.

$f_1 = u_1u_3 - u_3u_1$, $f_2 = u_1u_4 - u_4u_1 + u_2u_3 - u_3u_2$.

Definerende systemer:

$$\alpha_{u_2u_4} = \begin{cases} (0\ 1) \\ 0 \end{cases}, \quad \alpha_{u_4u_2} = \begin{cases} (0\ -1) \\ 0 \end{cases}$$

Tredje ordens MP:

$$\alpha_{u_2u_4}\alpha_{e_1} = 1, \quad \alpha_{u_2u_4}\alpha_{e_2} = x, \quad \alpha_{u_4u_2}\alpha_{e_1} = -1, \quad \alpha_{u_4u_2}\alpha_{e_2} = -x$$

$$f_1 = u_1u_3 - u_3u_1 - u_2u_4u_1 + u_4u_2u_1, \quad f_2 = u_1u_4 - u_4u_1 + u_2u_3 - u_3u_2 - u_2u_4u_2 + u_4u_2u_2$$

Resten av relasjonene $\equiv 0$. $H(M) \cong k\langle u_1,u_2,u_3,u_4\rangle/(f_1,f_2)$.

<u>Eks. 2</u>: $A = k[x,y]$, $M_1 = A/(x,y)$, $M_2 = A/(x^2,y)$.

(I) Beregning av $\mathrm{Ext}^i_A(M_r,M_s)$, $i=1,2$. Alle trivielle, unntatt $\mathrm{Ext}^i_A(M_1,M_2)$:

$$\begin{tikzcd}
0 & M_1 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x\ y)"'] & A \arrow[l, "\binom{-y}{x}"'] & 0 \arrow[l]
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(M_1,M_2) \arrow[r] & M_2 \arrow[r, "\binom{x}{0}"] & M_2^2 \arrow[r, "(0\ x)"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

$$(\bar p, \bar q) \in M_2^2$$

$$(0\ x)\begin{pmatrix}\bar p \\ \bar q\end{pmatrix} = x\bar q = 0 \iff xq \in (x^2,y) \iff xq = rx^2 + sy \iff xq - rx^2 = sy$$

$$\Rightarrow s = s_1x. \quad xq = rx^2 + s_1xy \Rightarrow q = rx + s_1y.$$

$$(\bar p, \bar q) = (\bar p, \overline{rx+s_1y}) = \bar p(1,0) + \bar r(0,x). \qquad \binom{x}{0}\cdot \bar a = \bar a(x\ 0).$$

k-basis blir altså $(1,0), (0,x)$.

<u>Yonedarepresentasjon</u>

<u>(1,1)</u>

$$\begin{tikzcd}
0 & M_1 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x\ y)"'] \arrow[dl, shift right, "(1\ 0)"'] \arrow[dl, shift left, "(0\ 1)"] & A \arrow[l, "\binom{-y}{x}"'] \arrow[dl, shift right, "\binom{0}{1}"'] \arrow[dl, shift left, "\binom{-1}{0}"] & 0 \arrow[l] \\
0 & M_1 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x\ y)"] & A \arrow[l, "\binom{-y}{x}"] & 0 \arrow[l]
\end{tikzcd}$$

<u>(1,2)</u>

$$\begin{tikzcd}
0 & M_1 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x\ y)"'] \arrow[dl, shift right, "(1\ 0)"'] \arrow[dl, shift left, "(0\ x)"] & A \arrow[l, "\binom{-y}{x}"'] \arrow[dl, shift right, "\binom{0}{1}"'] \arrow[dl, shift left, "\binom{-1}{0}"] & 0 \arrow[l] \\
0 & M_2 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x^2\ y)"] & A \arrow[l, "\binom{-y}{x^2}"] & 0 \arrow[l]
\end{tikzcd}$$

(2,1)

$$
\begin{tikzcd}
0 & M_2 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x^2\ y)"'] \arrow[dl, "(1\ 0)"'] \arrow[dl, shift left, "(0\ 1)"] & A \arrow[l, "\binom{-y}{x^2}"'] \arrow[dl, "\binom{0}{1}"'] \arrow[dl, shift left, "\binom{-x}{0}"] & 0 \arrow[l] \\
0 & M_1 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x\ y)"] & A \arrow[l, "\binom{-y}{x}"] & 0 \arrow[l]
\end{tikzcd}
$$

(2,2)

$$
\begin{tikzcd}
0 & M_2 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x^2\ y)"'] \arrow[dl, "(1\ 0)"'] \arrow[dl, shift left, "(x\ 0)"] \arrow[dl, shift left=2, "(0\ 1)"] \arrow[dl, shift left=3, "(0\ x)"] & A \arrow[l, "\binom{-y}{x^2}"'] \arrow[dl, "\binom{0}{1}"'] \arrow[dl, shift left, "\binom{0}{x}"] \arrow[dl, shift left=2, "\binom{-1}{0}"] \arrow[dl, shift left=3, "\binom{-x}{0}"] & 0 \arrow[l] \\
0 & M_2 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "(x^2\ y)"] & A \arrow[l, "\binom{-y}{x^2}"] & 0 \arrow[l]
\end{tikzcd}
$$

1. Ordens GMMP: Cup-produktene

(1,1)

$t_{11}(1)t_{11}(2) = -1,\ t_{11}(2)t_{11}(1) = 1,\ t_{12}(1)t_{21}(2) = -x = 0,\ t_{12}(2)t_{21}(1) = x = 0$

(1,2)

$t_{11}(1)t_{12}(2) = -1,\ t_{11}(2)t_{12}(1) = 1,\ t_{12}(1)t_{22}(3) = -1,\ t_{12}(1)t_{22}(4) = -x = 0,\ t_{12}(2)t_{22}(1) = x = 0,$

$t_{12}(2)t_{22}(2) = x^2 = 0$

(2,1)

$t_{21}(1)t_{11}(2) = -1,\ t_{21}(2)t_{11}(1) = 1,\ t_{22}(1)t_{21}(2) = -x = 0,\ t_{22}(2)t_{21}(2) = -x^2 = 0,$

$t_{22}(3)t_{21}(1) = 1,\ t_{22}(4)t_{21}(1) = x = 0$

(2,2)

$t_{22}(1)t_{22}(3) = -1,\ t_{22}(1)t_{22}(4) = -x,\ t_{22}(2)t_{22}(3) = -x,\ t_{22}(2)t_{22}(4) = -x^2 = 0,$

$t_{22}(3)t_{22}(1) = 1,\ t_{22}(3)t_{22}(2) = x,\ t_{22}(4)t_{22}(1) = x,\ t_{22}(4)t_{22}(2) = x^2 = 0,\ t_{21}(1)t_{12}(2) = -1,$

$t_{21}(2)t_{12}(1) = 1.$

2. Ordens relasjoner

$f_{11}(1) = t_{11}(2)t_{11}(1) - t_{11}(1)t_{11}(2)$

$f_{12}(1) = t_{11}(2)t_{12}(1) - t_{11}(1)t_{12}(2) - t_{12}(1)t_{22}(3)$

$f_{21}(1) = t_{21}(2)t_{11}(1) - t_{21}(1)t_{11}(2) + t_{22}(3)t_{21}(1)$

$f_{22}(1) = t_{22}(3)t_{22}(1) - t_{22}(1)t_{22}(3) + t_{21}(2)t_{12}(1) - t_{21}(1)t_{12}(2)$

$f_{22}(2) = t_{22}(3)t_{22}(2) - t_{22}(2)t_{22}(3) + t_{22}(4)t_{22}(1) - t_{22}(1)t_{22}(4)$

$$S_3 = \begin{pmatrix} k\langle t_{11}(1), t_{11}(2)\rangle & \langle t_{12}(1), t_{12}(2)\rangle \\ \langle t_{21}(1), t_{21}(2)\rangle & k\langle t_{22}(1), t_{22}(2), t_{22}(3), t_{22}(4)\rangle \end{pmatrix} \Big/ (f_{ij}(k))$$

$$B_2 = B_2' \setminus \{ t_{11}(2)t_{11}(1), t_{11}(2)t_{12}(1), t_{21}(2)t_{11}(1), t_{22}(3)t_{22}(1), t_{22}(4)t_{22}(1) \}$$

<u>2. ordens definerende systemer</u>

$$\alpha(t_{12}(1)t_{21}(2)) = \begin{cases} (0\ 1) \\ 0 \end{cases}, \quad \alpha(t_{12}(2)t_{21}(1)) = \begin{cases} (0\ -1) \\ 0 \end{cases}, \quad \alpha(t_{12}(1)t_{22}(4)) = \begin{cases} (0\ 1) \\ 0 \end{cases},$$

$$\alpha(t_{12}(2)t_{22}(1)) = \begin{cases} (0\ -1) \\ 0 \end{cases}, \quad \alpha(t_{12}(2)t_{22}(2)) = \begin{cases} (0\ -x) \\ 0 \end{cases}, \quad \alpha(t_{22}(2)t_{22}(4)) = \begin{cases} (0\ 1) \\ 0 \end{cases}$$

$$\alpha(t_{22}(4)t_{22}(2)) = \begin{cases} (0\ -1) \\ 0 \end{cases}, \quad \alpha(t_{22}(1)t_{21}(2)) = \begin{cases} 0 \\ \binom{1}{0} \end{cases}, \quad \alpha(t_{22}(4)t_{21}(1)) = \begin{cases} 0 \\ \binom{-1}{0} \end{cases},$$

$$\alpha(t_{22}(2)t_{21}(2)) = \begin{cases} 0 \\ \binom{x}{0} \end{cases}$$

$I_3 = \underline{m}^3/\underline{m}^4 + \underline{m}(f_{ij}(k)) \cap \underline{m}^3$, $\underline{m} = \underline{\text{rad}}$. $B_3' \subseteq \{ \underline{t} \mid \underline{t} = t_{ij}(k)\underline{u},\ \underline{u} \in B_2 \}$.

<u>Tredje ordens MP.</u>

$t_{12}(1)t_{21}(2)t_{11}(1) = 1$, $t_{12}(1)t_{21}(2)t_{12}(1) = 1$, $t_{12}(2)t_{21}(1)t_{11}(1) = -1$, $t_{12}(2)t_{21}(1)t_{12}(1) = -1$,

$t_{12}(1)t_{22}(4)t_{21}(1) = 1$, $t_{12}(1)t_{22}(4)t_{22}(1) = 1$, $t_{12}(1)t_{22}(4)t_{22}(2) = x = 0$,

$t_{12}(2)t_{22}(1)t_{21}(1) = -1$, $t_{12}(2)t_{22}(1)t_{22}(1) = -1$, $t_{12}(2)t_{22}(1)t_{22}(2) = -x = 0$

$t_{12}(2)t_{22}(2)t_{21}(1) = -x = 0$, $t_{12}(2)t_{22}(2)t_{22}(1) = -x = 0$, $t_{12}(2)t_{22}(2)t_{22}(2) = -x^2 = 0$,

$t_{22}(2)t_{22}(4)t_{21}(1) = 1$, $t_{22}(2)t_{22}(4)t_{22}(1) = 1$, $t_{22}(2)t_{22}(4)t_{22}(2) = x$,

$t_{22}(4)t_{22}(2)t_{21}(1) = -1$, $t_{22}(4)t_{22}(2)t_{22}(1) = -1$, $t_{22}(4)t_{22}(2)t_{22}(2) = -x$, $t_{12}(1)t_{22}(1)t_{21}(2) = 1$,

$t_{22}(1)t_{22}(1)t_{21}(2) = 1$, $t_{22}(2)t_{22}(1)t_{21}(2) = x = 0$, $t_{12}(1)t_{22}(4)t_{21}(1) = -1$, $t_{22}(1)t_{22}(4)t_{21}(1) = -1$,

$t_{22}(2)t_{22}(4)t_{21}(1) = -x = 0$, $t_{12}(1)t_{22}(2)t_{21}(2) = x = 0$, $t_{22}(1)t_{22}(2)t_{21}(2) = x = 0$, $t_{22}(2)t_{22}(2)t_{21}(2) = x^2 = 0$.